ex: a) for the curve $r = 1+\sin\theta$ find those points at which tan line is horizontal or vertical for $0 \le \theta \le 2\pi$.

b) sketch the curve for $0 \le \theta \le 2\pi$.

Sol'n: a) we know $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

potential horiz. tan line when $\dfrac{dy}{dx} = 0$

i.e. $\dfrac{dr}{d\theta}\sin\theta + r.\cos\theta = 0$

but $r = 1+\sin\theta$ so $\dfrac{dr}{d\theta} = \cos\theta$

so we want: $\cos\theta\sin\theta + (1+\sin\theta)\cos\theta = 0$

$\Rightarrow 2\cos\theta\sin\theta + \cos\theta = 0$

$\Rightarrow \cos\theta(2\sin\theta+1) = 0$

$\Rightarrow \cos\theta = 0$ or $2\sin\theta + 1 = 0$

$\Rightarrow \theta = \pi/2, 3\pi/2$ $\quad\Downarrow$

$\sin\theta = -\frac{1}{2}$

$\theta = 7\pi/6, 11\pi/6$

at these θ's we have:

$r(\pi/2) = 1+\sin(\pi/2) = 2$ $\quad r(7\pi/6) = 1+\sin(7\pi/6) = 1/2$

$r(3\pi/2) = 1+\sin(3\pi/2) = 0$ $\quad r(11\pi/6) = 1/2$

potential vert. tan lines when

$\frac{dy}{dx}$ "=" $\pm\infty$

Solve for: $\frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$

i.e. $\cos^2\theta - (1+\sin\theta)\sin\theta = 0$

$\Rightarrow \cos^2\theta - \sin\theta - \sin^2\theta = 0$

$\Rightarrow (1-\sin^2) - \sin\theta - \sin^2\theta = 0$

$\Rightarrow 1 - \sin\theta - 2\sin^2\theta = 0$

"$1 - x - 2x^2 = 0$"

$\Rightarrow (1-2\sin\theta)(1+\sin\theta) = 0$

$\Rightarrow \sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = -1$

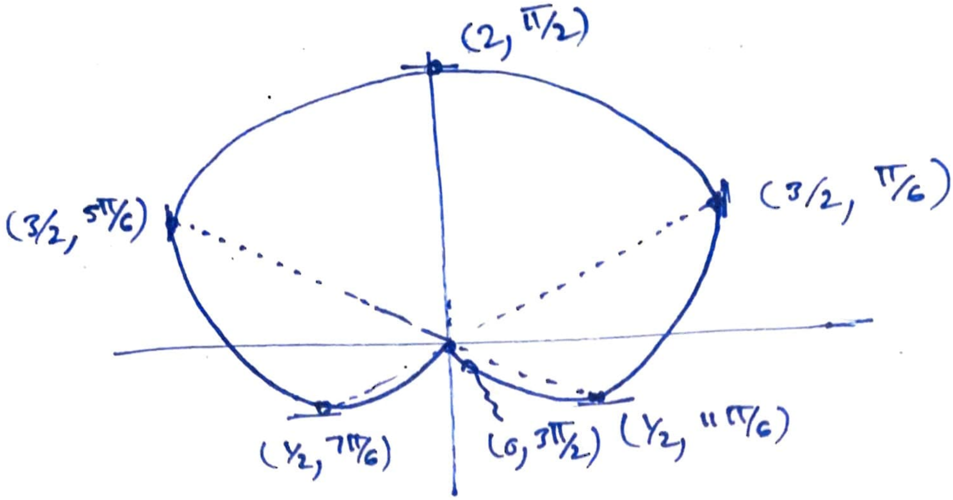
$\theta = \pi/6, 5\pi/6$ $\qquad$ $3\pi/2$

at these points:

$r(3\pi/2) = 1 + \sin(3\pi/2) = 0$

$r(\pi/6) = 1 + \sin(\pi/6) = \frac{3}{2} = r(5\pi/6)$

Let's graph all these points:



Observe: @ $(1/2, 7\pi/6)$, $(1/2, 11\pi/6)$, $(2, \pi/2)$

have horiz. tan lines since

$$\frac{dy}{dx} = \frac{0}{\text{nonzero}}$$

@ $(3/2, 5\pi/6)$, $(3/2, \pi/6)$,

vert. tan lines since

$$\frac{dy}{dx} = \frac{\text{nonzero}}{0}$$

but @ $(0, 3\pi/2)$ unclear

since $\frac{dy}{dx} = \frac{0}{0}$

We use L'Hospital:

$$\lim_{\theta \to 3\pi/2} \frac{dy}{dx} = \lim_{\theta \to 3\pi/2} \frac{(\cos\theta)(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$ (24)

$$= \left[\lim_{\theta \to \frac{3\pi}{2}} \left(\frac{\cos\theta}{1+\sin\theta}\right)\right]\left[\lim_{\theta \to 3\pi/2} \frac{1+2\sin\theta}{1-2\sin\theta}\right]$$

$$\underset{\frac{0}{0}}{\parallel} \qquad\qquad \underset{\frac{-1}{3}}{\parallel}$$

$$= -\frac{1}{3} \lim_{\theta \to 3\pi/2} \frac{\cos\theta}{1+\sin\theta}$$

$$\overset{L'H}{=} -\frac{1}{3} \lim_{\theta \to 3\pi/2} \frac{-\sin\theta}{\cos\theta} = -\frac{1}{3}(\infty) = -\infty$$

So there is a vertical tan line

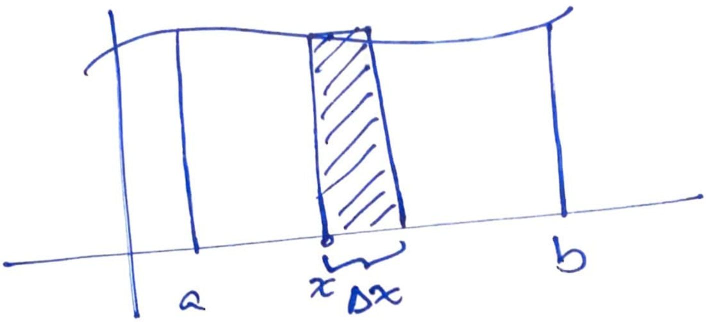
@ this point.

10.4 Integration and length in polar coordinates

Recall: integration in rectangular coords:

to find area under $y = f(x)$ between $x = a$ and $x = b$ first:

approximate:



segment area $\approx f(x)\,\Delta x$

total area:

$$\approx \sum f(x)\Delta x$$

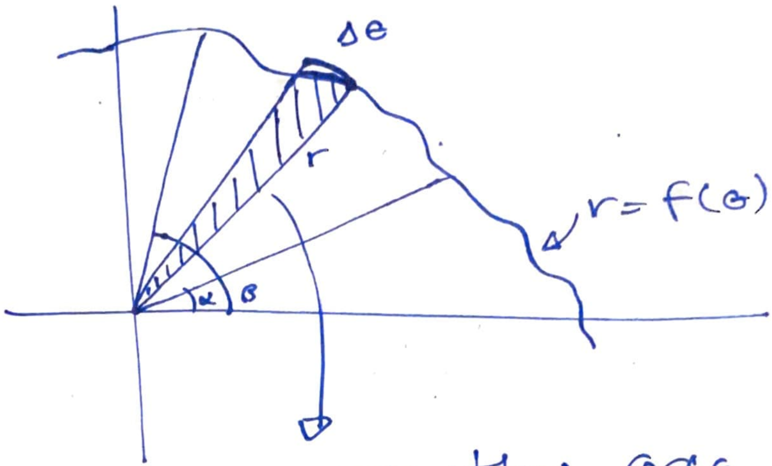
then take limit:

$$\text{area} = \int_a^b f(x)\,dx$$

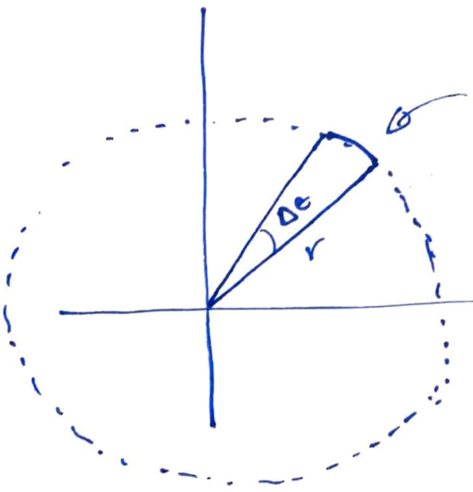
$$= \int_a^b y\,dx$$

In polar coords:

to find area bounded by polar curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$, first approximate:



approx this area by a circular sector:



sector area
$= \frac{1}{2} r^2 \Delta\theta$

why: total circle area is:

$\frac{1}{2} r^2 (2\pi)$

$= \pi r^2$