Taylor and Maclaurin Series

The big Q: which functions f have power series rep'ns? How do we find them?

So far: started w/

$$f(x) = \frac{1}{1-x} = \sum_{0}^{\infty} x^n \quad \text{for } -1<x<1$$

and found power series rep'ns for variations on this function: $\frac{1}{1+x^2}$, $\ln(1+x)$, etc.

A new approach: Sps we are given a function $f(x)$ and we assume $f(x)$ has a power series rep'n:

$$f(x) = \sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

on some interval of the form $(a-R, a+R)$

But we don't know the c_n's:

How do we find them?

Observe: by our differentiation rules for power series, we have:

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

so: $f'(a) = c_1 = 1!\,c_1$

$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4\cdot 3\overset{c_4}{\vee}(x-a)^2 + \cdots$

so: $f''(a) = 2c_2 = 2!\,c_2$

$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3(x-a)^2 + \cdots$

so: $f'''(a) = 3\cdot 2\, c_3 = 3!\,c_3$

and in general we see:

$$f^{(n)}(x) = n!\,c_n + \text{terms w/ } (x-a)$$

so: $f^{(n)}(a) = n!\,c_n$

We get a formula for c_n:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This is big news: if we can find derivatives $f^{(n)}(a)$, we can solve for the coefficients c_n in f's power series rep'n (assuming such a rep'n exists)

We've proved:

Thm: If $f(x)$ has a power series rep'n centered at a, i.e. if there is $R>0$ such that:

$$f(x) = \sum_{0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then the coefficients c_n are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

so that:

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } |x-a|<R$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- this series is called the Taylor series for $f(x)$ centered @ a.
- in the case when $a=0$, called the Maclaurin series for f.

Thm says:- $\{$ IF $\}$ f(x) can be represented by a power series @ $x=a$, then that power series is given by the Taylor series.

- some f's do not have a power series rep'n anywhere; such f's will not equal their Taylor series anywhere.
- we will typically ignore the question: "does this f(x) have a Taylor series expansion @ $x=a$?" and just assume that it does.
- But you should be aware this assumption is being made.
  (see book for a way to prove a given f has a Taylor series)

Ex: Find the Maclaurin series for $f(x) = e^x$ and its radius of convergence.

Soln: By thm, Maclaurin series given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In this case $f(x) = e^x$.

so $f'(x) = f''(x) = f'''(x) = \dots = e^x$

Hence $f^{(n)}(0) = e^0 = 1$ for every n.

So Maclaurin series is:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

we've checked before: this series converges everywhere (i.e. radius is ∞)

can be proved: e^x equals its Maclaurin series, i.e.,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

for every x.

In particular:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Can use this expression to get approx'ns for e.

for example: $1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}$
$= 2.708\bar{3}$

whereas $e = 2.718\ldots$

A given function $f(x)$ can have different power series rep's around different centers a.

ex: Find the Taylor series for e^x around $a=2$.

soln: by theorem, Taylor series given by:

$$\sum_0^\infty \frac{f^{(n)}(2)}{n!}(x-2)^n$$

here $f(x) = e^x$ so $f^{(n)}(x) = e^x$ so $f^{(n)}(2) = e^2$ for every n.

Taylor series is:

$$\sum_{n=0}^\infty \frac{e^2}{n!}(x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \ldots$$

ez to check: series converges everywhere and can prove is $= e^x$ everywhere.

Hence $e^x = \sum_0^\infty \frac{e^2}{n!}(x-2)^n$ for every x