ex: same Q for $\ln(1+x)$

soln: a new trick.

Observe: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$= \frac{1}{1-(-x)}$

geo series fmla $= \sum_{n=0}^{\infty} (-x)^n = \sum_{0}^{\infty} (-1)^n x^n$

$= 1 - x + x^2 - x^3 + \cdots$

for $|-x| < 1$

$|x| < 1$

Hence: $\ln(1+x) = \int \sum_{0}^{\infty} (-1)^n x^n \, dx$

$= \int (1 - x + x^2 - x^3 + \cdots) \, dx$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + C$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + C$, if $|x| < 1$

To solve for C let $x = 0$.

$\ln(1) = 0 + C$

i.e. $0 = C$ Hence: $\ln(1+x) = \sum_{1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

if $|x| < 1$.

ex: same Q for $f(x) = \tan^{-1}(x)$

Sol'n: we know:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_0^\infty (-x^2)^n \qquad \text{if } |-x^2|<1$$

i.e. $|x|<1$

$$= \sum_0^\infty (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

so: $\tan^{-1}x = \int 1 - x^2 + x^4 - x^6 + \cdots$ for $|x|<1$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + C$$

to find c, let $x=0$

$$\tan^{-1}0 = 0 + C$$

i.e. $0 = C$

so $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

$$= \sum_0^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$$ ✓

ex: evaluate $\int \frac{x}{1+x^3}\,dx$ as a power series. What is radius of convergence?

Sol'n: First we do $\frac{x}{1+x^3}$

we know: $\frac{x}{1+x^3} = x\left(\frac{1}{1+x^3}\right)$

$= x\left(\frac{1}{1-(-x^3)}\right)$

$= x \sum_0^\infty (-x^3)^n$ if $|-x^3|<1$ i.e. $|x|<1$

$= x\left(\sum_0^\infty (-1)^n x^{3n}\right) = x(1-x^3+x^6-x^9+\dots)$

$= \sum_0^\infty (-1)^n x^{3n+1}$

$= x - x^4 + x^7 - x^{10} + \dots$ for $|x|<1$

Hence: $\int \frac{x}{1+x^3}\,dx$

$= \int x - x^4 + x^7 - x^{10} + \dots \, dx$ for $|x|<1$

$= \frac{1}{2}x^2 - \frac{1}{5}x^5 + \frac{1}{8}x^8 - \frac{1}{11}x^{11} + \dots + C$

$= \sum_0^\infty (-1)^n \frac{x^{3n+2}}{3n+2} + C$ for $|x|<1$

Note: it turns out:

$$\int \frac{x}{1+x^3}\,dx = \frac{1}{6}\Big(-\ln(x^2-x+1) + 2\ln(x+1) + 2\sqrt{3}\tan^{-1}\Big(\frac{2x-1}{\sqrt{3}}\Big)\Big) + C$$

but this is hard to solve for directly.

Our power series rep'n of this integral is good enough for many applications.

Ex: Approximate $\int_0^{0.3} \frac{x}{1+x^3}\,dx$ using the previous power series, to w/in 2 decimal places.

sol'n: from above:

$$\int \frac{x}{1+x^3}\,dx = \frac{1}{2}x^2 - \frac{1}{5}x^5 + \frac{1}{8}x^8 - \cdots + C$$

$$= \sum_0^\infty (-1)^n \frac{x^{3n+2}}{3n+2} + C \quad \text{for } |x|<1$$

Hence: $$\int_0^{0.3} \frac{x}{1+x^3}\,dx = \Big(\frac{1}{2}(.3)^2 - \frac{1}{5}(.3)^5 + \frac{1}{8}(.3)^8 - \cdots\Big) \cancel{+C}$$

$$- \Big(\cancel{\frac{1}{2}0^2 - \frac{1}{5}0^5 + \frac{1}{8}0^8 - \cdots}\Big)$$

$$= \frac{1}{2}(.3)^2 - \frac{1}{5}(.3)^5 + \frac{1}{8}(.3)^8 - \cdots$$

- this is a convergent alternating series (175)
- let's approx. it, using first 3 terms:

$$\approx \frac{1}{2}(.3)^2 - \frac{1}{5}(.5)^2 + \frac{1}{8}(.3)^8$$

$$= .044522220125..$$

we know error is bounded by next terms:

$$R_2 \le b_{2+1} = b_3 = \frac{1}{11}(.3)^{11} = 1.61 \times 10^{-7}$$

i.e. .0445 ... is w/in 1.61×10^{-7} of

$$\int_0^{0.3} \frac{x}{1+x^3}\,dx$$

Way better than 2 decimal places!