Ex: Consider the curve C defined by $x = t^2$
$y = t^3 - 3t$

(a) Show that C has two tangents at the point $(3,0)$ and find their slopes.

soln: observe: $y = 0 \Rightarrow t^3 - 3t = 0$
$\Rightarrow t(t^2 - 3) = 0$
$\Rightarrow t = 0$ or $t = \sqrt{3}$
or $t = -\sqrt{3}$

$\hookrightarrow$ curve crosses x-axis at these t's.

at $t = 0$, $x = 0^2 = 0$
$t = \sqrt{3}$, $x = (\sqrt{3})^2 = 3$
$t = -\sqrt{3}$, $x = (-\sqrt{3})^2 = 3$

So: curve hits $(3,0)$ at $t = \pm\sqrt{3}$.
It crosses itself at this point.

we have: $dy/dt = y'(t) = 3t^2 - 3$
$dx/dt = x'(t) = 2t$

So: $dy/dx = \dfrac{3t^2 - 3}{2t}$

$\to$ at $t = \sqrt{3} \to 6/2\sqrt{3} = \sqrt{3} \approx 1.7$

$\to$ at $t = -\sqrt{3} \to \dfrac{6}{-2\sqrt{3}} \approx -1.7$

(b) Find the points on the curve w/ horizontal or vertical tangent lines. (224)

$\frac{dy}{dx} = 0$ $\qquad\qquad$ $\frac{dy}{dx} = \pm\infty$

Sol'n: horizontal: we solve $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{3t^2-3}{2t} = 0$$

$$\Rightarrow 3t^2 - 3 = 0 \Rightarrow t^2 - 1 = 0$$
$$\Rightarrow (t-1)(t+1) = 0$$
$$\Rightarrow t = \pm 1$$

@ $t = 1$: $x = 1^2 = 1$, $y = 1^3 - 3\cdot 1 = -2$

@ $t = -1$: $x = (-1)^2 = 1$, $y = (-1)^3 - 3(-1) = 2$

So we have horizontal tangents @ $(1,-2)$ and $(1,2)$

vertical: solve: $\frac{dy}{dx} = \pm\infty$ i.e.

i.e. $\frac{3t^2-3}{2t} = \pm\infty \Rightarrow 2t = 0$
$\Rightarrow t = 0$

really I mean:

$$\lim_{t\to 0} \frac{3t^2-3}{2t} = -\infty$$

@ $t = 0$: $x = 0^2 = 0$, $y = \underbrace{0^3 - 3\cdot 0}_{=0}$

So: vertical tangent @ $(0,0)$

Second derivatives: we can again reason using chain rule to get a formula for $\frac{d^2y}{dx^2}$ in terms of t's:

Chain rule says:

$$\frac{d}{dt}(\ldots) = \frac{d}{dx}(\ldots)\frac{dx}{dt}$$

so: $$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}_{\frac{dy^2}{dx^2}} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$ ta da!

(c) Find where the curve C is concave up and concave down

$\frac{d^2y}{dx^2} \geq 0$ $\qquad$ $\frac{d^2y}{dx^2} \leq 0$

soln: First we find $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$= \frac{\dfrac{2t(6t) - (3t^2-3)2}{(2t)^2}}{2t}$$

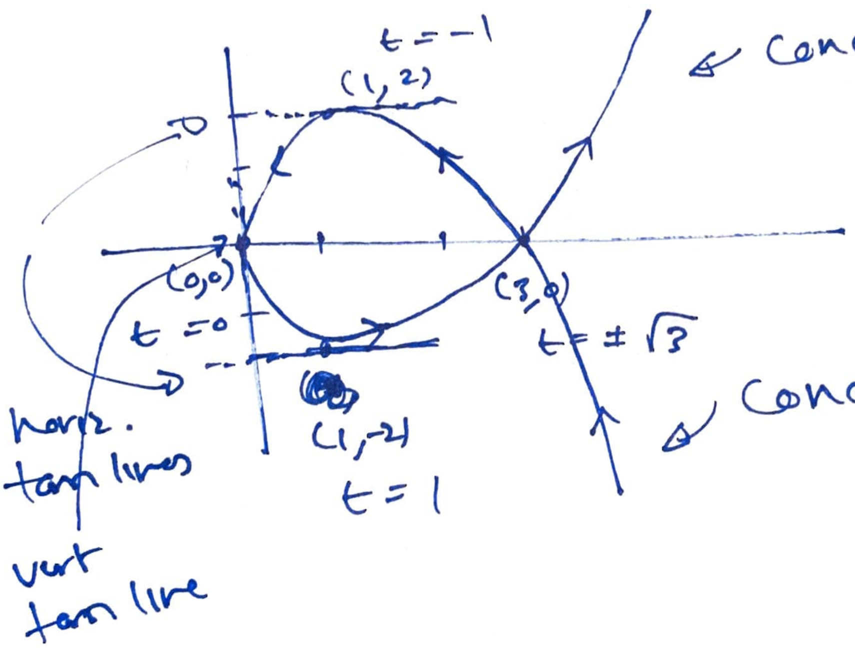
$$= \frac{12t^2 - 6t^2 + 6}{(2t)^3}$$

$$= \frac{6t^2+6}{8t^3} = \frac{3t^2+3}{4t^3}$$

since numerator always positive, second deriv is
>0 when $4t^3>0$
<0 when $4t^3<0$

i.e. >0 when $t>0$
<0 when $t<0$

(d) we can now sketch curve:



Concave up $t>0$

Concave down $t<0$

Arc Length:

Theorem If a curve C is parametrized by:

$$x = f(t)$$
$$y = g(t)$$

and this parametrization does not overlap itself (except perhaps at isolated points) for $\alpha \le t \le \beta$ then the length of C over $\alpha \le t \le \beta$ is given by:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

↳ formula can be derived in a similar way to our previous arc length formula:

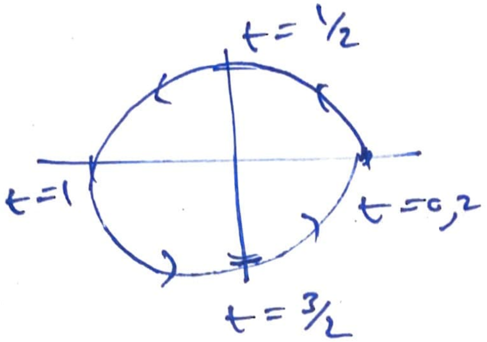
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ for $y = f(x)$ over $a \le x \le b$

↳ see book for details.

ex: Find the circumference of the unit circle using the parametrization

$$x = \cos(\pi t)$$
$$y = \sin(\pi t)$$

Sol'n: this parametrization traverses the circle exactly once as t goes from 0 to 2: (228)



we have $\frac{dx}{dt} = -\pi \sin(\pi t)$

$\frac{dy}{dt} = \pi \cdot \cos(\pi t)$

so by our formula:

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t)}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 (\sin^2(\pi t) + \cos^2(\pi t))}\, dt$$

$$= \int_0^2 \sqrt{\pi^2}\, dt$$

$$= \int_0^2 \pi = \pi t \Big|_0^2 = 2\pi \checkmark$$

notice: if we instead integrated from $t=0$ to $t=4$ we get:

$$\int_0^4 \pi \, dt$$

$$= 4\pi$$

→ this corresponds to the length of the path traversing the circle <u>twice</u>