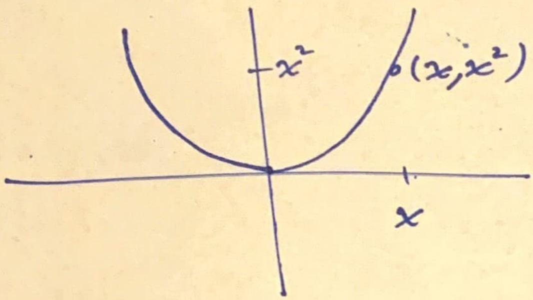
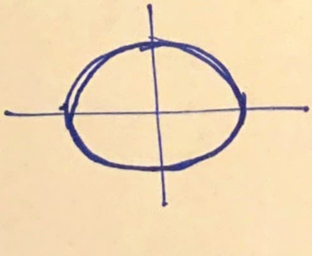
Ch. 10: Calculus of curves

- Given a continuous function $f(x)$, graph of $y = f(x)$ is a curve in xy-plane.
- consists of all points (x,y) s.t. $y = f(x)$
- e.g. graph of $y = x^2$ consists of all points of the form (x, x^2) in the plane:



- not all curves are of this form.
- e.g. the unit circle is not the graph of any function (fails VLT) but can still be described by eq'n $x^2 + y^2 = 1$



graph consists of all points (x,y) s.t. $x^2+y^2=1$.

- Another way to define curves: define both coords x and y as functions of a third variable t.

$$x = x(t)$$
$$y = y(t)$$

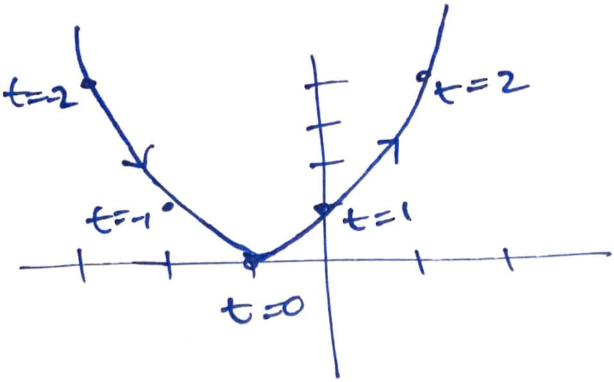
called a <u>parametrized curve.</u>

- think of t as a time variable: @ time t a particle is at $(x(t), y(t))$ and sweeps out a curve as time moves forward.

<u>Ex</u>: define $x = t-1$, $y = t^2$. sketch the parametrized curve.

<u>Soln</u>: one way: plot some points!

t	x	y
0	-1	0
1	0	1
2	1	4
3	2	9
-1	-2	1
-2	-3	4
-3	-4	9

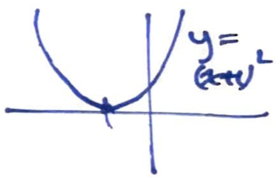


Looks like: parabola w/ vertex @ $(-1,0)$

<u>Notice</u>: w/ param. curves there is a <u>direction</u>: in this case from left to right as t increases

another way: "eliminate the parameter" t to get eq'n in x and y describing curve.

$$x = t - 1 \Rightarrow t = x+1$$
$$y = t^2 \Rightarrow y = (x+1)^2$$



This work shows: <u>if</u> (x,y) is a pt on curve <u>then</u> it must lie on the parabola $y=(x+1)^2$. Not necessarily every point on parabola is obtained (though in this case this happens)

<u>Note</u>: in passing to eq'n $y=(x+1)^2$ we lose info about direction of flow as t increases.

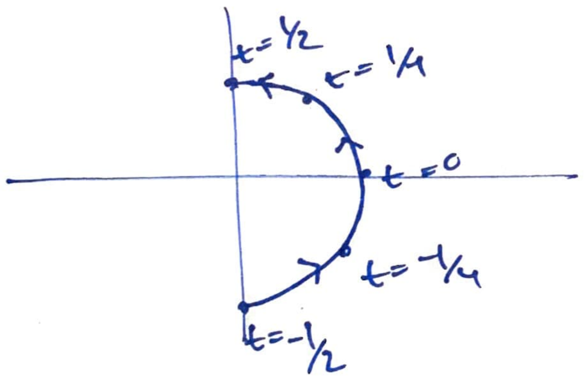
ex: graph curve: $x = \cos(\pi t)$
$y = \sin(\pi t)$

① for t in $[-1/2, 1/2]$
② for all t

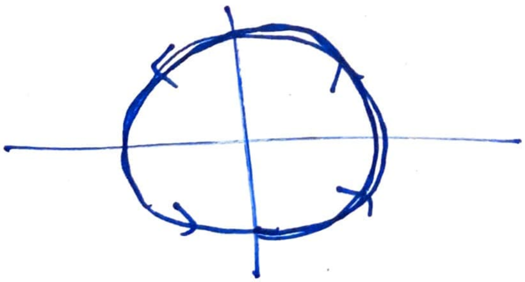
Sol'n: ①

t	x	y
$-1/2$	0	-1
$-1/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
0	1	0
$1/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$1/2$	0	1

$x = \cos(-\pi/2) = 0$
$y = \sin(-\pi/2) = -1$



② continuing as $t \to \infty$ we see:



in this case if we eliminate parameter:

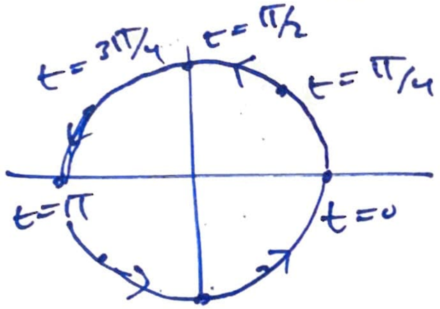
$$x^2+y^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1$$

So our curve lies on the circle $x^2+y^2=1$ (as we've seen) ✓

notice: elim'ing param. not always as ez as solving for t

ex: a curve can have more than one parametrization.

Consider: $x = \cos(t) \quad y = \sin(t)$



Same graph as above but curved traversed more slowly as t increases

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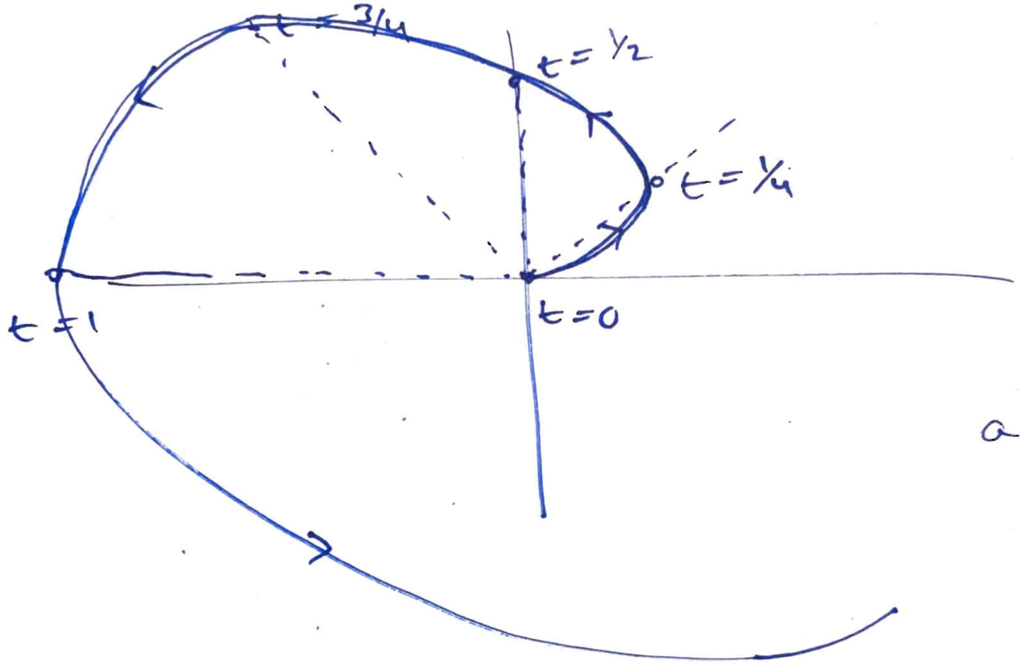
more generally: graph of $x = r\cos t$, $y = r\sin t$, where $r>0$ is fixed, is a circle of radius r centered @ (0,0)

ex: graph $x = t \cdot \cos(\pi t)$
$y = t \cdot \sin(\pi t)$

we know: $(r\cos\theta, r\sin\theta)$ is the pt on a circle of radius r @ angle θ.

so for us: @ time t point is $(t\cos\pi t, t\sin\pi t)$

so our "radius" is t angle is πt.



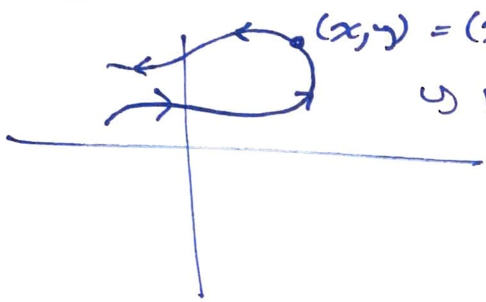
a spiral!

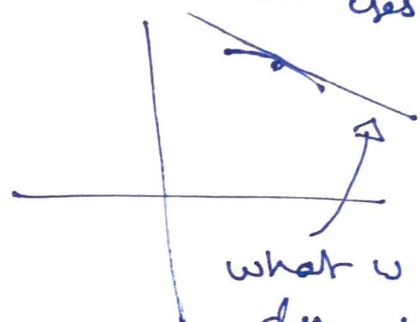
note: not easy (or useful) to eliminate parameter in this case.

Calculus of parametric curves

- We've seen: a parametric curve $x = x(t)$ $y = y(t)$ need not be the graph of any function (can fail VLT)

- however: if we only look at a piece of the curve near a pt $(x,y) = (x(t_0), y(t_0))$ it may look like a function of x locally.





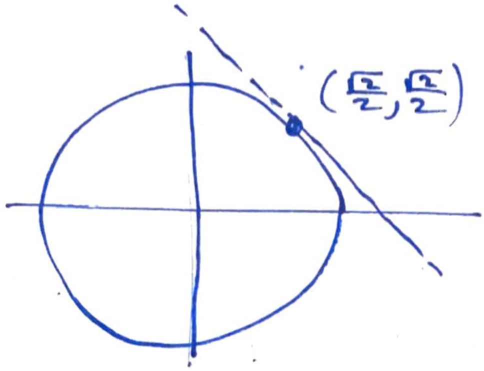
- at such a point, still makes sense to ask: what is slope of tan line, i.e. what is dy/dx?

- if we think of $y = y(x)$ as a function of x locally, then since y and x both functions of t, we have by chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \leftarrow \text{as long as } dx/dt \neq 0$$

ex: Find the slope of the tan line to the circle $x = \cos(t)$, $y = \sin(t)$ at the pt $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

soln:



notice: $x = y = \frac{\sqrt{2}}{2}$ when $t = \pi/4$

slope is: $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$\frac{dy}{dt} = \cos t$ $\quad \frac{dx}{dt} = -\sin(t)$

at $t = \pi/4$ we have: $\frac{dy}{dt} = \cos(\pi/4) = \sqrt{2}/2$

$\frac{dx}{dt} = -\sin(\pi/4) = -\sqrt{2}/2$

So: $\frac{dy}{dx} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$ ✓

Ex: Consider the curve C defined by $x = t^2$
$y = t^3 - 3t$

(a) Show that C has two tangents at the point $(3,0)$ and find their slopes.

Sol'n: observe: $y = 0 \Rightarrow t^3 - 3t = 0$
$\Rightarrow t(t^2 - 3) = 0$
$\Rightarrow t = 0$ or $t = \sqrt{3}$
or $t = -\sqrt{3}$

↳ curve crosses x-axis at these t's.

at $t = 0$, $x = 0^2 = 0$
$t = \sqrt{3}$, $x = (\sqrt{3})^2 = 3$
$t = -\sqrt{3}$, $x = (-\sqrt{3})^2 = 3$

So: curve hits $(3,0)$ at $t = \pm\sqrt{3}$.
It crosses itself at this point.

we have: $dy/dt = y'(t) = 3t^2 - 3$
$dx/dt = x'(t) = 2t$

so: $dy/dx = \dfrac{3t^2 - 3}{2t}$

→ at $t = \sqrt{3} \rightarrow 6/2\sqrt{3} = \sqrt{3} \approx 1.7$

→ at $t = -\sqrt{3} \rightarrow \dfrac{6}{-2\sqrt{3}} \approx -1.7$