if: ① $\lim_{n\to\infty} b_n = 0$ and ② $b_{n+1} \le b_n$ for every n

then: $\sum_{1}^{\infty} (-1)^n b_n$ converges.

e.g. $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

converges since ① $\lim_{n\to\infty} \frac{1}{n} = 0$ and ② $\frac{1}{n+1} < \frac{1}{n}$

Also useful: Alt. series estimation: if $\sum (-1)^n b_n$ satisfies ① and ② then:

$$S_n = b_1 - b_2 + b_3 - \cdots \pm b_n$$

is w/in b_{n+1} of entire series $\sum (-1)^{n+1} b_n$

(i.e. $|R_n| \le b_{n+1}$)

e.g. $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = .8\bar{3}$ is w/in $\frac{1}{4}$ of $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{n}$ (in fact $\sum (-1)^{n+1} \frac{1}{n} = \ln(2) = .693\ldots$)

(vi) ratio and root tests:

Def'n: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

Fact: absolute convergence implies convergence (but not other way around) ex: Sps $\sum_0^\infty a_n$ is a series, and consider $\sum_0^\infty |a_n|$. Let $S_n = |a_0| + |a_1| + \cdots + |a_n|$ be the nth partial sum. Assume $S_n \le 1$ for all n.

Does $\sum a_n$ converge?

A: yes! the sequence S_n is:

- increasing: since $S_{n+1} = S_n + |a_{n+1}| \geq S_n$
- bounded (by 1, by assumption)

Hence S_n converges by monotone convergence

i.e. $\lim_{n\to\infty} S_n = L$ exists.

Hence by def'n, $\sum |a_n|$ converges

Hence by fact, $\sum a_n$ converges.

Note: ratio/root tests give absolute convergence.

e.g. $\sum_1^\infty (-1)^{n-1} \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$

converges absolutely, since $\lim_{n\to\infty} \left| \frac{(-1)^n \, 1/(n+1)!}{(-1)^{n-1} \, 1/n!} \right|$

$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

as does $\sum_{1}^{\infty} \frac{1}{n^n}$ since

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1.$

<u>Power Series</u>: a power series (centered @ $x=a$) is written:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

e.g. $\sum_{0}^{\infty} (-1)^n/(n+1) \; x^n = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \cdots$

is a power series.

A power series is <u>not</u> a series. Becomes a series if we specify x.

e.g. if $x=2$ in above we get:

$$\sum_{0}^{\infty} \frac{(-1)^n}{n+1} 2^n = 1 - \frac{1}{2}\cdot 2 + \frac{1}{3}\cdot 2^2 - \frac{1}{4} 2^3 + \cdots$$

(diverges)

<u>The question</u>: for which x's does $\sum_{0}^{\infty} c_n(x-a)^n$ converge? Use ratio/root test to answer.

e.g. for series above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{((-1)^{n+1}/n+2)\, x^{n+1}}{((-1)^n/n+1)\, x^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} |x| = |x|$$

By ratio test: series converges if $|x|<1$ (11)
diverges if $|x|>1$ and if $x=1,-1$ have to check by hand (you try)

On its interval of convergence, a power series defines a function:

$$\sum_0^\infty c_n(x-a)^n = f(x) \quad \text{on} \quad (a-R, a+R)$$

Can think of $f(x)$ as an "infinite polynomial." Differentiation and integration behave as we expect.

if $f(x) = \sum_0^\infty c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

then $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_1^\infty n c_n (x-a)^{n-1}$

and $\int f(x)\,dx = c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots + C$
$= \sum_0^\infty c_n \frac{(x-a)^{n+1}}{n+1} + C$

Amazing fact: many well-known functions have power series rep'ns (at least on parts of their domain)

$$\frac{1}{1-x} = \sum_0^\infty x^n \qquad x \text{ in } (-1,1)$$

$$e^x = \sum_0^\infty \frac{1}{n!}x^n \qquad \text{for all } x$$

$$e^{-x^2} = \sum_0^\infty \frac{1}{n!}(-x^2)^n = \sum_0^\infty \frac{(-1)^n}{n!}x^{2n} \qquad \text{all } x$$

We can use these rep'ns to get others:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_0^\infty x^n \qquad x \text{ in } (-1,1)$$

$$= \frac{d}{dx}(1+x+x^2+\cdots)$$

$$= 1+2x+3x^2+\cdots$$

$$= \sum_1^\infty n x^{n-1} \qquad x \text{ in } (-1,1)$$

Can even get power series rep'n for functions we can't otherwise write:

$$\int e^{-x^2}dx = \int \sum_0^\infty \frac{(-1)^n}{n!}x^{2n} = \sum_0^\infty \frac{(-1)^n}{n!(2n+1)}x^{2n+1} + C$$

can't be written in terms of elem. functions

<u>Taylor series</u>: if $f(x)$ has a power series rep'n @ $x=a$, it's given by its Taylor series:

$$f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

e.g. if $f(x) = \sin(x)$

$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1$$
$$\vdots$$

So Taylor series @ $a=0$:

$$\sin(x) = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$
$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$= \sum_0^\infty \frac{1}{(2n+1)!} x^{2n+1}$$ (converges for all x to $\sin(x)$)

Power series rep'ns useful for estimating functions $f(x)$ by polynomials.

if $f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $(a-R, a+R)$

the Nth Taylor polynomial is:

$$\sum_0^N \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

we have: $f(x) \approx T_N(x)$ on $(a-R, a+R)$
where $\approx$ gets better the bigger the N and the closer x is to a.

Error given by: $R_N(x) = \sum_{N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Can sometimes bound $R_N(x)$ w/ alt. series estimation thm.

e.g. $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$

by alt. series est. thm

$x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is w/in $\left|\frac{x^7}{7!}\right|$ of $\sin(x)$.

So e.g. in $(-1,1)$ this poly approx's $\sin(x)$ to w/in $\frac{1}{7!}$

If alt series doesn't apply, use Taylor's theorem to bound $R_N(x)$.