ex: determine int. of conv. for:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$$

soln $a_n = \left(\frac{x-2}{n}\right)^n$

easier to use root test here:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x-2}{n}\right|^n}$$

$$= \lim_{n\to\infty} \left|\frac{x-2}{n}\right|$$

$$= 0 \text{ , no matter what } x \text{ is}$$

by root test, series converges for all x i.e. interval of convergence is $(-\infty, \infty)$

Power Series as functions

Idea: can view a power series

$$\sum_{0}^{\infty} c_n (x-a)^n$$

as a function $f(x)$ on its interval of convergence.

amazingly: many well-known functions can be represented as power series!

We already know one example:

$$\text{if } |x|<1 \text{ then } f(x) = \frac{1}{1-x}$$

$$\text{is equal to } = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots$$

$$\boxed{= \sum_{1}^{\infty} 1 x^{n-1} = \frac{1}{1-x} \text{ by geo. series fmla}}$$

- of course, if $|x| \geq 1$, $\sum x^n$ diverges, so is no longer equal to $\frac{1}{1-x}$

- but we'll see: representing functions $f(x)$ on even parts of their domain can be useful!

ex: express $f(x) = \frac{1}{1+x^2}$ as a power series and find the interval of convergence.

Sol'n: we know $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$

as long as $|t|<1$.

So let $t=-x^2$ we get:

$$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1-x^2+x^4-x^6+\cdots$$

as long as $|-x^2|<1$, i.e. $x^2<1$, i.e. $|x|<1$. Because $\sum_0^{\infty}(-x^2)^n$ is a geometric series (with $r=-x^2$), this is as good as we can do: so, series converges if and only if $|-x^2|<1$

i.e. interval of convergence is $(-1,1)$.

ex: same question for $f(x) = \frac{1}{x+2}$

sol'n: $\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$

$$= \frac{1}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)^n x^n$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 - \dots\right)$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \dots$$

$$= \sum_{n=0}^{\infty}(-1)^n(1/2)^{n+1}x^n$$

if and only if

converging iff $\left|-\frac{x}{2}\right| < 1$

i.e. $\left|\frac{x}{2}\right| < 1$

i.e. $-1 < \frac{x}{2} < 1$

i.e. $-2 < x < 2$

so interval of convergence in this case $(-2, 2)$.

ex: same question for $f(x) = \frac{x^3}{x+2}$

soln:- we know we can distribute constants over convergent series: $c \sum a_n = \sum c a_n$

- we just showed, for a fixed x in $(-2,2)$, the series $\sum (-1)^n (\frac{1}{2})^{n+1} x^n$ converges to $\frac{1}{x+2}$.

- so for x in $(-2,2)$ we must have:

$$x^3 \sum_0^\infty (-1)^n (\tfrac{1}{2})^{n+1} x^n = \sum_0^\infty x^3 (-1)^n (\tfrac{1}{2})^{n+1} x^n$$

$$= \sum_0^\infty (-1)^n (\tfrac{1}{2})^{n+1} x^{n+3}$$

converges to $\frac{x^3}{x+2}$!

so: $\sum_0^\infty (-1)^n (\frac{1}{2})^{n+1} x^{n+3} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \dots$

is a power series representation of $\frac{x^3}{x+2}$ in the interval $(-2,2)$

- Recall: we think of a power series

$$\sum_0^\infty c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

as an "infinite polynomial"

- polynomials are nice because they are ez to differentiate and integrate.

- turns out: same is true of power series!

Theorem If the power series $\sum_0^\infty c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by:

$$f(x) = \sum_0^\infty c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is continuous and differentiable on $(a-R, a+R)$

Further: (i) $f'(x) = \sum_1^\infty n c_n (x-a)^{n-1}$

$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(ii) $\int f(x)\,dx = \sum_0^\infty c_n \frac{(x-a)^{n+1}}{n+1} + C$

$$= \left(c_0(x-a) + \frac{c_1 (x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots\right) + C$$

and the radius of convergence for both series is also R.

- won't prove – beyond scope
- theorem says: can differentiate/integrate power series term by term, just like polynomials!
- also says: radius R of convergence for differentiated + integrated series is same as orig. series, but what happens at $x = a+R$ and $x = a-R$ might be different.
- Can use theorem to get power series rep'ns for new functions from old ones.

ex: Find a power series rep'n for $\frac{1}{(1-x)^2}$. What is radius of convergence?

Sol'n: we know:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{if } |x| < 1$$

R.

so: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}(1+x+x^2+\cdots)$

$$\frac{1}{(1-x)^2} \overset{\text{by theorem}}{=} 1+2x+3x^2+\cdots$$
$$= \sum_{1}^{\infty} n x^{n-1} = \sum_{0}^{\infty} (n+1)x^n$$

thm also says: $R=1$ for this series too.

can check: $\sum (n+1)x^n$ diverges if $x=1$ or $x=-1$

So interval of convergence is still $(-1,1)$

in this case: interval same as original, but in general can be different.

ex: Find a power series rep'n for $\frac{x}{(1-x)^2}$.

Sol'n: by above:

$$\frac{1}{(1-x)^2} = \sum_{0}^{\infty}(n+1)x^n = 1+2x+3x^2+\cdots$$

if $|x|<1$

so: $$\frac{x}{(1-x)^2} = x\sum_{0}^{\infty}(n+1)x^n = x(1+2x+3x^2+\cdots)$$
$$= x+2x^2+3x^3+\cdots$$
$$= \sum_{0}^{\infty}(n+1)x^{n+1} \text{ for } |x|<1$$

mult. by x does not change radius.