# Ch. 11 Review

Sequences: a sequence $a_n$ is an infinite list of numbers

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ $\quad (a_n = \frac{1}{2^n})$

$3, 5, 7, \ldots$ $\quad (a_n = 2n+1)$

$1, -1, 1, -1, \ldots$ $\quad (a_n = (-1)^{n+1})$

are sequences. Usually start @ $n=1$.

$\lim_{n\to\infty} a_n = L$ means the terms $a_n$ get arbitrarily close to $L$ as $n$ gets larger. (formal def'n in terms of $\varepsilon$'s)

say: $a_n$ converges to $L$ if this happens.

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ converges to $0$

$3, 5, 7, \ldots$ diverges (to $\infty$)

$1, -1, 1, -1 \ldots$ diverges (oscillates)

# Showing convergence/divergence of sequences

(1) limit laws: e.g. consider sequence

$$a_n = \frac{n^2+1}{2n^2+n} \quad \left(\frac{2}{3}, \frac{5}{10}, \frac{10}{21}, \dots\right)$$

remember: fmla just a description for a list of numbers.

then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{2n^2+n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)} = \dots \quad \text{(some more steps)}$$

$$= \frac{1}{2}$$

(2) theorem: if $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$

then $\lim_{n\to\infty} a_n = L$ too

(often combine w/ L'hospital)

by thm, assuming lim exists

e.g. $\lim_{n\to\infty} n \cdot \sin(1/n) = \lim_{x\to\infty} x\sin(1/x)$

$$= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \quad \text{“}\frac{0}{0}\text{”}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{-\cancel{\frac{1}{x^2}}\cos(1/x)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \cos(1/x) = \cos\left(\lim_{x\to\infty} \frac{1}{x}\right)$$
$$= \cos(0) = 1 \checkmark$$

(3) <u>inspection</u>:

e.g. if $a_n = (-1)^{n+1}$ $(1, -1, 1, -1, \ldots)$
then $\lim_{n\to\infty} a_n$ DNE since sequence oscillates.

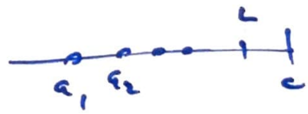
(4) <u>monotone convergence</u>:

used sometimes to show $\lim_{n\to\infty} a_n$ <u>exists</u> (w/o computing it)

<u>Thm</u>: Given $a_n$ a sequence:

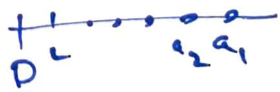
if either

① $a_1 \le a_2 \le a_3 \le \ldots$ and there is $C$ s.t. $C \ge a_n$ for all $n$;




<u>or</u>:

② $a_1 \ge a_2 \ge \ldots$ and there is $D$ s.t. $D \le a_n$ for all $n$




<u>then</u>: $\lim_{n\to\infty} a_n$ exists.

## Series:

- a **series** is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- e.g. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

- the nth partial sum is defined as:

$$S_n = a_1 + a_2 + \cdots + a_n$$

- e.g. for $\sum_{n=1}^{\infty} \frac{1}{2^n}$ we have:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\vdots$$

<u>Note</u> $S_1, S_2, S_3, \ldots$ is a <u>sequence</u>, the terms of which we view as better and better approximating $\sum_{1}^{\infty} a_n$

- if this sequence has a limit, i.e. if $\lim_{n\to\infty} S_n = L$ then we say the series $\sum_{n=1}^{\infty} a_n$ converges to $L$ and write

$$\sum_{n=1}^{\infty} a_n = L.$$

- e.g. for $\sum_{n=1}^{\infty} \frac{1}{2^n}$ we saw: $S_n = \frac{2^n - 1}{2^n}$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n-1}{2^n}$
$= \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$

so: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots = 1$

- Given a series $\sum_{n=1}^{\infty} a_n$ can also consider the sequence of terms $a_n$

- e.g. for $\sum_{n=1}^{\infty} \frac{1}{2^n}$, the sequence of terms is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

- Finding the sum $\sum_{1}^{\infty} a_n$ is about finding $\lim_{n\to\infty} S_n$, not $\lim_{n\to\infty} a_n$.

- But there is a relationship between $\lim a_n$ and $\sum a_n$

<u>Thm</u> (Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_1^\infty a_n$ diverges (i.e. $\lim_{n\to\infty} S_n$ diverges)

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e.g. $\sum_1^\infty \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots$

diverges because $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

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<u>Tests for convergence of $\Sigma a_n$:</u>

(i) Geo. series test: $\sum_1^\infty ar^{n-1}$ converges iff $|r|<1$

if and only if

e.g. $\sum_1^\infty 2(\frac{1}{3})^{n-1} = 2 + \frac{2}{3} + \frac{2}{9} + \cdots$ converges: $\frac{1}{3} < 1$

$\sum_1^\infty 2\cdot 3^{n-1} = 2 + 6 + 18 + \cdots$ diverges: $3 \geq 1$

(ii) p-series test: $\Sigma \frac{1}{n^p}$ converges iff $p > 1$

e.g.: $\sum_1^\infty \frac{1}{n^2}$ converges

$\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges

(iii) integral test: if $a_n = f(n)$ where $f(x)$ is positive and decreasing on $[1,\infty)$ then:

$\sum_1^\infty a_n$ converges iff $\int_1^\infty f(x)\,dx$ converges

e.g. $\sum_2^\infty \frac{1}{n\sqrt{\ln(n)}}$ diverges since

$\int_2^\infty \frac{1}{x\sqrt{\ln(x)}}\,dx = \infty$ (see: midterm 2)

(iv) Comparison: applies to series $\sum a_n$ $\sum b_n$ w/ positive terms.

Direct: ex's: • $\sum_1^\infty \frac{n}{n^3+1}$ converges since

$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$ and $\sum_1^\infty \frac{1}{n^2}$ converges

• $\sum_1^\infty \frac{n^2+1}{n^3}$ diverges since

$\frac{n^2+1}{n^3} > \frac{n^2}{n^3} = \frac{1}{n}$ and $\sum_1^\infty \frac{1}{n}$ diverges.

limit: if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $0<L<\infty$

then $\sum_1^\infty a_n$ converges iff $\sum_1^\infty b_n$ converges

• $\sum_2^\infty \frac{n}{n^3-1}$ converges because:

$\lim_{n\to\infty} \frac{n/(n^3-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1$ and $\sum \frac{1}{n^2}$ converges

(v) Alt. series test: Given $\sum_1^\infty (-1)^n b_n$ where $b_n \geq 0$: