tho if we zoom out, approx'ns get worse.

Big Q: can we quantify how well $T_N(x)$ approx's $e^x$?

Need:

Def'n: Sps $f(x) = \sum_0^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $(a-R, a+R)$. Define the nth remainder:

$$R_N(x) = f(x) - T_N(x)$$

$$= \sum_{N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \frac{f^{(N+2)}(a)}{(N+2)!}(x-a)^{N+2}$$

$$+ \cdots$$

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- For a given $x$ in $(a-R, a+R)$ $R_N(x)$ tells us how far $T_N(x)$ is from $f(x)$
- can we bound $R_N(x)$?

**Taylor's Theorem** Sps $\sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $(a-R, a+R)$.

If there is a constant $M$ such that $|f^{(N+1)}(x)| \leq N$ on an interval $(a-d, a+d)$ for some $d \leq R$

Then: $|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1}$ on $(a-d, a+d)$

Says: if we can bound the $(N+1)$st derivative of $f$ on an interval $(a-d, a+d)$ around $a$, then we can bound size of remainder on that interval.
↳ won't prove.

ex: ① Approximate $f(x) = \sqrt[3]{x} = x^{1/3}$ by $T_2(x)$ around $a = 8$.
② How accurate is this approx'n for $x$ in $(7, 9)$?

Sol'n: ① Let's find $T_2(x)$

$$= f(8) + f'(8)(x-8) + f''(8)\frac{(x-8)^2}{2!}$$

$f(8) = \sqrt[3]{8} = 2$

$f'(x) = \frac{1}{3} x^{-2/3}$ , $f'(8) = \frac{1}{3} 8^{-2/3} = \frac{1}{12}$

$f''(x) = -\frac{2}{9} x^{-5/3}$ , $f''(8) = -\frac{2}{9} 8^{-5/3}$

$= -\frac{1}{144}$

$f'''(x) = \frac{10}{27} x^{-8/3}$ (will need to bound $R_2(x)$)

so: $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

$\approx \sqrt[3]{x}$ near $a = 8$.

② By Taylor's thm, if we can find $M$ s.t. $|f'''(x)| \leq M$ on $(7, 9)$ then we can bound $R_2(x)$ on this

<u>Observe</u>: $x \geq 7$ on $(7, 9)$

so: $x^{8/3} \geq 7^{8/3}$ " "

so: $\frac{1}{x^{8/3}} \leq \frac{1}{7^{8/3}}$ " "

so: $f'''(x) = \frac{10}{27} \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}}$ on $(7, 9)$

$\approx .0021$

Taylor says:

$$|R_2(x)| \leq \frac{.0021}{3!}|x-8|^3 \quad \text{for } x \text{ in } (7,9)$$

$$\leq \frac{.0021}{3!} 1^3 \quad \text{" "}$$

$$= .0004$$

This means: given $x$ in $(7,9)$:

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

is w/in $.0004$ of $\sqrt[3]{x}$.

for example: $T_2(8.3) = 2.0246875\ldots$

$\sqrt[3]{8.3} = 2.024693\ldots$

Sometimes we can use alt. series estimation theorem instead of Taylor's thm to bound $R_N(x)$.

ex: we can approx. $\sin(x) = \sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

by $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (near $x = 0$)

what is max error in this approx'n for $x$ in $[-.3, .3]$?

soln: since the Maclaurin series for $\sin(x)$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating for any fixed $x$, the diff. between

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

and the entire series is at most $\left|\frac{x^7}{7!}\right|$

If $x$ is in $[.3, .3]$, i.e. $|x| \leq .3$, we have

$$\left|\frac{x^7}{7!}\right| \leq \left|\frac{.3^7}{7!}\right| \approx 4.3 \times 10^{-8}$$

So: $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is quite a good approx'n to $\sin(x)$ on $[-.3, .3]$

e.g. $\sin(\pi/15) = .20791169081\ldots$

.2094...

whereas: $\left(\frac{\pi}{15}\right) - \frac{(\pi/15)^3}{3!} + \frac{(\pi/15)^5}{5!} = .207911694\ldots$

first place of difference