ex: Find the Maclaurin series for $f(x) = \sin(x)$ and its radius of convergence

soln: series given by:

$$\sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

we check:

$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1$$
$$f^{(4)}(0) = \sin(0) = 0$$
$$f^{(5)}(0) = \cos(0) = 1$$

$\vdots$ etc.

So series is:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To find radius, use ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \Big/ \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \frac{(2n+1)!}{(2n+3)!} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \frac{1}{(2n+2)(2n+3)} \right|$$

$$= x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = 0, \quad \text{for every } x$$

$\Rightarrow$ series converges everywhere. ($R = \infty$)

Can be proven: $\sin(x)$ = its Maclaurin series where it converges

Hence: $\sin(x) = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all $x$.

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ex: Find the Maclaurin series for $f(x) = \cos(x)$.

Sol'n: Two approaches:

→ could use def'n $\sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n$

or → differentiate series for $\sin(x)$

Let's try the second.

$$\cos(x) = \frac{d}{dx} \sin(x) = \frac{d}{dx} \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_0^\infty \frac{d}{dx} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ (by diff'e thm)

$$= \sum_0^\infty (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)!}$$

$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Since series of $\sin(x)$ converges for all $x$, series for $\cos(x)$ does as well.

<u>Hence</u>: $\cos(x) = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$

<u>for all $x$</u>

<u>ex</u>: Find Maclaurin series for:

$f(x) = (1+x)^k$, where $k$ is any real number.

<u>sol'n</u>: ez to check:

$f(0) = 1$ and

$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$ for $n \geq 1$

So Maclaurin series is:

$$\sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n = \sum_0^\infty \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Can check: radius of convergence is 1 (use ratio test).

Can be proved: $(1+x)^k =$ its Maclaurin series on its interval of convergence.

Hence: for $|x|<1$,

$$(1+x)^k = \sum_0^\infty \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

when $n=0$, coefficient is 1

the coefficient $\frac{k(k-1)\cdots(k-n+1)}{n!}$ is often denoted $\binom{k}{n}$.

So can rewrite:

$$= \sum_0^\infty \binom{k}{n} x^n$$

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This series is mainly useful when $k$ is fractional, or negative.

See table 1 pg. 768 for list of useful Taylor/Maclaurin series.

ex: Find Maclaurin series for

$f(x) = \frac{1}{\sqrt{1+x}}$

Soln: $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$

$= \sum_0^\infty \binom{-1/2}{n} x^n$ by above, for $|x|<1$.

$= \sum_0^\infty \frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!} x^n$

$= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} x^3 + \cdots$

$= \sum_0^\infty (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n n!} x^n$

Converges on $(-1,1)$