Let $L = \lim_{n\to\infty} a_n$.

We've shown $L = 3 - \frac{1}{L}$

$$\Rightarrow \quad L^2 = 3L - 1$$

$$\Rightarrow \quad L^2 - 3L + 1 = 0$$

$$\Rightarrow \quad L = \frac{3 \pm \sqrt{5}}{2}$$

which of $\frac{3+\sqrt{5}}{2}$ $\frac{3-\sqrt{5}}{2}$ is our actual limit?

must be this, since a_n is increasing and $a_2 = 2 > \frac{3-\sqrt{5}}{2}$

So: $\lim_{n\to\infty} a_n = \frac{3+\sqrt{5}}{2} \approx 2.618\ldots$

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note: first equality above is just the fact that if the sequence

$a_1, a_2, a_3, \ldots$ converges to L

then $a_2, a_3, \ldots$ converges to L also

i.e. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

11.2 Series

Given a sequence a_n, the associated series is the "infinite sum"

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

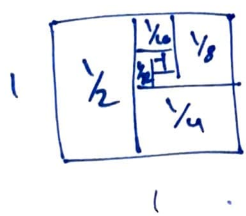
→ does this concept make sense?
→ is such a sum always $= \infty$?

ex: Consider the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$\vdots = \infty$? not so fast.

THE UNIT SQUARE



area $= 1 \times 1 = 1$

Seems like: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$= \text{area of square}$

$= 1.$

To make precise, need to define convergence of a series.

Def'n: Given a series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums s_n defined by:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

- if: $\lim_{n\to\infty} s_n = L$, we say the series $\sum_{n=1}^{\infty} a_n$ converges to L and write $\sum_{n=1}^{\infty} a_n = L$.

- if $\lim_{n\to\infty} s_n$ does not exist (or $= \pm\infty$) we say $\sum_{n=1}^{\infty} a_n$ diverges.

ex: for the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

the first few partial sums are:

$$s_1 = a_1 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$S_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\vdots$

$$S_n = \frac{2^n - 1}{2^n}$$

can compute: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n}$

$$= \lim_{n\to\infty} 1 - \frac{1}{2^n}$$

$$= 1$$

Hence series converges to 1

Write: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

ex: the series

$$1+1+1+\cdots = \sum_{n=1} 1$$

does not converge, since in this case; $S_n = \underbrace{1+1+1+\cdots+1}_{n \text{ times}} = n$

so $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$

Def'n a geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

when a, r are real numbers.

e.g.

$$\sum_{n=1}^{\infty} 5(1/3)^{n-1} = 5 + 5/3 + 5/9 + 5/27 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$\sum_{n=1}^{\infty} 3 \cdot 2^{n-1} = 3 + 6 + 12 + 24 + \cdots$$

are geometric series.

Thm: The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

converges if and only if $|r| < 1$.

In this case, we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Pf: assume $|r|<1$ (If $|r|\geq 1$, it's clear series diverges).

nth partial sum given by:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \dots + ar^{n-1}$$

$$= a(1 + r + \dots + r^{n-1})$$

$$\Rightarrow (1-r)S_n = a(1 + r + \dots + r^{n-1})(1-r)$$

$$= a(1 + \cancel{r} + \cancel{r^2} + \dots + \cancel{r^{n-1}} - \cancel{r} - \cancel{r^2} - \cancel{r^3} - \dots - \cancel{r^{n-1}} - r^n)$$

$$= a(1-r^n)$$

So: $$S_n = \frac{a(1-r^n)}{1-r}$$

hence: $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-\cancel{r^n}^{\,0})}{1-r} \quad \text{since } |r|<1$$

$$= \frac{a}{1-r} \checkmark$$

we've shown $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$