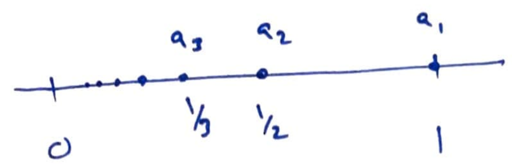
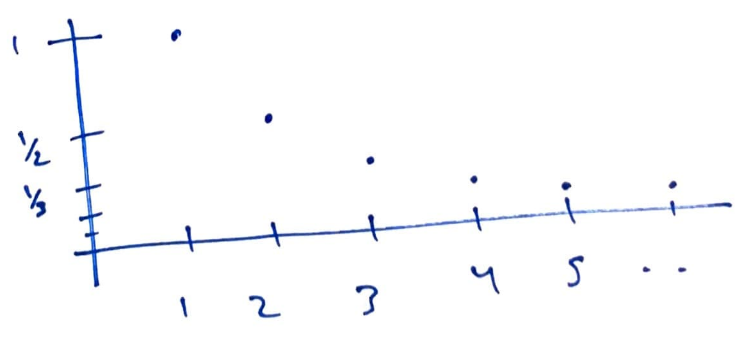
Given a sequence a_n (e.g. $a_n = \frac{1}{n}$) 90.5
will typically visualize 1-dimensionally:



Can also visualize "functionally"
(i.e. thinking of a_n as a function on
the positive integers)



Notation: will use "a_n" to refer
to both the nth term in a
sequence, and the entire sequence
itself.

- Sequences, like functions, can have limits
- the notation

$$\lim_{n\to\infty} a_n = L$$

intuitively means:

"as n gets larger, a_n gets arbitrarily close to L"

E.g. consider the sequence

$$1/2, 2/3, 3/4, 4/5, \ldots \quad \left(a_n = \frac{n}{n+1}\right)$$

Intuitively for this sequence we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

To prove this, need a rigorous def'n of limit

Def'n: We say the sequence a_n has limit L, and write

$$\lim_{n\to\infty} a_n = L$$

if:

For every $\varepsilon > 0$
there is an integer N
such that for every $n > N$
we have $|a_n - L| < \varepsilon$

"no matter how close I want to get to L"
"I can go out far enough in the sequence"
"such that at any later point"
"I'm at least that close to L"

<u>Ex</u>: Let $a_n = \frac{n}{n+1}$ be the sequence above. Then $\lim_{n \to \infty} a_n = 1$.

<u>Pf</u>: - Fix $\varepsilon > 0$

- Let N be an integer large enough so that $\frac{1}{N} < \varepsilon$
- then for any $n > N$, observe we have: $\frac{1}{n} < \frac{1}{N}$
- Hence, for any $n > N$ we have:

$$\begin{aligned} |a_n - 1| &= \left|\frac{n}{n+1} - 1\right| \\ &= \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| \\ &= \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} \\ &< \frac{1}{n} < \frac{1}{N} < \varepsilon \end{aligned}$$

Since ε was arbitrary (i.e. we could have repeated the same argument for any ε) the claim is proved ✓

Summary: we showed no matter how small ε is to begin with, we can find a term in our sequence a_N, such that all subsequent terms a_n are within ε of l. Hence l is the limit. ✓

Terminology: - if $\lim_{n\to\infty} a_n = L$ we say a_n converges to L.

- if a_n does not have a limit we say the sequence diverges.

↳ we can also define what it means for a sequence a_n to have a limit of ∞ or $-\infty$.

$\lim_{n\to\infty} a_n = \infty$ means "as n gets larger and larger, a_n becomes arbitrarily large."

$\lim_{n\to\infty} a_n = -\infty$ means "as n gets larger, a_n becomes arbitrarily negative"

↳ see book for formal def'ns.

In practice we use theorems that allow us to compute limits w/o using ε def'n.

Theorem (limit laws) Sps a_n and b_n are convergent sequences and c is a constant. Then:

- $\lim_{n\to\infty} c = c$ ← i.e. the limit of the sequence whose every term is c, is c.
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} a_n \cdot b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ as long as $\lim_{n\to\infty} b_n \neq 0$

- if $f(x)$ is a continuous function

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) \qquad (*)$$

- in <u>particular</u>: if p is a fixed number:

$$\lim_{n\to\infty} (a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p$$

(why: $f(x) = x^p$ is continuous)

Ex: <u>Claim</u>: $\lim_{n\to\infty} \frac{n}{n+1} = 1$

<u>Pf</u>: already verified w/ lim def'n, but easier w/ limit laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

Another useful fact:

Thm: Suppose $f(x)$ is a function and $\lim_{x\to\infty} f(x) = L$.

If we define a sequence by
$$a_n = f(n)$$

Then: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$ also.

↳ thm allows us to exploit L'hospital's rule to compute limits of certain sequences

ex: Consider the sequence defined by $a_n = \dfrac{n^2}{e^{n^2}}$.

$$\frac{1}{e}, \frac{4}{e^4}, \frac{9}{e^9}, \ldots$$

Then: $\lim_{n\to\infty} \dfrac{n^2}{e^{n^2}} = 0$

Pf: $\lim_{n\to\infty} \dfrac{n^2}{e^{n^2}} = \lim_{x\to\infty} \dfrac{x^2}{e^{x^2}}$ $\dfrac{\infty}{\infty}$

by theorem

$= \lim_{x\to\infty} \dfrac{2x}{2xe^{x^2}}$

L'hospital

$= \lim_{x\to\infty} \dfrac{1}{e^{x^2}} = 0$ ✓