For this series, $a_n = \frac{1}{n!} x^n$

so: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{1/(n+1)! \quad x^{n+1}}{1/n! \quad x^n} \right|$

$= \lim_{n \to \infty} \frac{n!}{(n+1)!} |x|$

$= \lim_{n \to \infty} \frac{1}{n+1} |x|$

$= 0$

no matter what x is!

by ratio test: this power series converges for <u>all</u> real numbers x!

<u>ex</u>: For which x does the power series $\sum_{0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$ converge?

<u>Sol'n</u>: for this series

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! \, x^{n+1}}{n! \, x^n} \right|$

$= \lim_{n \to \infty} (n+1) |x|$

$= \infty$ if $x \neq 0$.
$= 0$ if $x = 0$

By ratio test this series diverges unless $x = 0$.

- The power series we've seen so far are sometimes called power series centered @ 0

- can more generally consider expressions of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a is some real number.

↳ called the power series centered @ a.

Ex: For which values of x does the series $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converge?

$$\| $$

$$(x-3) + \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 + \cdots$$

Sol'n: in this case: $a_n = \frac{1}{n}(x-3)^n$

we check:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}/n+1}{(x-3)^n/n}\right|$$

$$= \lim_{n\to\infty}\left|(x-3)\frac{n}{\cancel{n}+1}\right|$$

$$= |x-3|$$

by ratio test, series converges if

$|x-3| < 1$, i.e. $-1 < x-3 < 1$

i.e. $2 < x < 4$.

also, series diverges if

$|x-3| > 1$, i.e if $x > 4$ or $x < 2$.

What if $x = 2$ or $x = 4$? Ratio test inconclusive, have to check by hand.

if $x = 4$, series becomes

$\sum_{n=1}^{\infty} \frac{1}{n}$ → diverges if $x = 2$, series becomes:

$$\sum_{1}^{\infty} (-1)^n \frac{1}{n}$$

Converges by alt. series test.

So overall: series converges if x in $[-2, 4)$ and diverges otherwise.

Notice: in our examples so far, the interval of x's for which we have convergence has the same center as the series (either 0, or 3)

This turns out to be true in general:

Theorem: For a power series $\sum_{0}^{\infty} c_n (x-a)^n$ exactly one of the following holds:

(i) series converges only if $x = a$

(ii) series converges for all x

(iii) there is a positive real number $R > 0$ such that series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

See book for proof.

(iii) says: x in $(a-R, a+R)$ $\hookrightarrow$ converge

in $(-\infty, a-R)$ or $(a+R, \infty)$ $\hookrightarrow$ diverge

if $x = a-R$ or $a+R$
can converge or diverge
have to check!

- R is called the <u>radius of convergence</u>
- in case (iii), 4 possibilities for the <u>interval of convergence</u> $(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R)$ $[a-R, a+R]$
- in case (ii) say radius is ∞
  interval is $(-\infty, \infty)$
- in case (i) say radius is 0
  interval is $x = a$

So far:

	radius	interval
$\sum_0^\infty x^n = 1 + x + x^2 + \cdots$	1	$(-1, 1)$
$\sum_0^\infty n!\,x^n = 1 + x + 2x^2 + 6x^3 + \cdots$	0	$x = 0$
$\sum_0^\infty \frac{x^n}{n!}$	∞	$(-\infty, \infty)$
$\sum_1^\infty \frac{(x-3)^n}{n}$	1	$[2, 4)$

ex: determine radius + interval of convergence (160)

for: $\sum_{0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}} x^n$

Sol'n: here $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$

we check: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1} x^{n+1}/\sqrt{n+2}}{(-3)^n x^n/\sqrt{n+1}}\right|$

$= \lim_{n\to\infty} 3|x| \underbrace{\sqrt{\frac{n+2}{n+1}}}_{\to 1}$

$= 3|x|$

by ratio test: if $3|x| < 1$, converges

i.e. $|x| < 1/3 \Rightarrow x \in (-1/3, 1/3)$

so radius is $1/3$.

what about endpoints?

if $x = 1/3$: series becomes

$\sum_{0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum \frac{(-1)^n}{\sqrt{n+1}}$

converges by alt. series test since

$\frac{1}{\sqrt{n+1}} \to 0$ and decreases if $x = -1/3$.

$$\sum_0^\infty \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}}$$

$$= \sum_0^\infty \frac{(1)^n}{\sqrt{n+1}} = \sum_0^\infty \frac{1}{\sqrt{n+1}}$$

diverges (do lim comparison w/ $\frac{1}{\sqrt{n}}$)

So: interval of convergence is $(-1/3, 1/3]$

ex: Find rad. + int. of conv. for

$$\sum_0^\infty \frac{n}{3^{n+1}} (x+2)^n$$

sol'n: here $a_n = \dfrac{n(x+2)^n}{3^{n+1}}$

we check: $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{\dfrac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\dfrac{n(x+2)^n}{3^{n+1}}} \right|$

$$= \lim_{n\to\infty} \left| (x+2) \frac{n+1}{n} \cdot \frac{1}{3} \right|$$

$$= \frac{1}{3} |x+2|$$

So series converges if:

$$\frac{1}{3}|x+2| < 1$$

$$\Rightarrow |x+2| < 3$$

$$\Rightarrow -3 < x+2 < 3$$

$$\Rightarrow -5 < x < 1$$

If $x = 5$, series is: $\sum_{0}^{\infty} \frac{n(-3)^n}{3^{n+1}}$

$$= \sum_{0}^{\infty} \frac{n}{3} \frac{(-3)^n}{3^n}$$

$$= \sum_{0}^{\infty} \frac{n}{3} (-1)^n$$

diverges by divergence test.

If $x = 1$, $\sum_{0}^{\infty} \frac{n\,3^n}{3^{n+1}}$

$$= \sum_{0}^{\infty} \frac{1}{3} n$$ also divergent.

So: radius is 3, interval is $(-5, 1)$