$$= \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}.$$

hence $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{\cancel{n+1}}{\cancel{n}}\right)^3 \frac{1}{3}$ 1

$$= \frac{1}{3} < 1.$$

by ratio test, $\sum (-1)^n \frac{n^3}{3^n}$ converges absolutely (hence converges).

Note: if we'd just wanted to show convergence, could've gotten away w/ alt. series test.

(2) What about $\sum_1^\infty \frac{n!}{100^n}$?

<u>Sol'n</u> in this case:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \frac{100^n}{100^{n+1}}\right|$$

$$= \left|\frac{n+1}{100}\right|$$

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{100} = \infty$

by ratio test, Series diverges.

(3) What about $\sum_{1}^{\infty} \frac{\sqrt{n}}{1+n^2}$?

Sol'n: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}\right|$

$$= \sqrt{\frac{n+1}{n}} \cdot \frac{1+n^2}{n^2+2n+2}$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \frac{1+n^2}{n^2+2n+2}$

$$= 1 \cdot 1$$

$\hookrightarrow$ ratio test inconclusive.

but observe:

$$\frac{\sqrt{n}}{1+n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

and $\sum \frac{1}{n^{3/2}}$

<u>converges</u>

By comparison:

$\sum \frac{\sqrt{n}}{1+n^2}$ <u>converges</u>

Thm (Root test):

Sps $\sum_1^\infty a_n$ is a series

(i) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum_1^\infty a_n$ converges absolutely

(ii) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum_1^\infty a_n$ diverges (includes if $L = \infty$)

(iii) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ or DNE, inconclusive.

- See book for proof (also uses geo. series)
- useful for dealing w/ series involving nth powers.

ex: Does $\sum_1^\infty \left(\frac{5n+6}{7n+20}\right)^n$ converge?

Sol'n in this case

$$\sqrt[n]{|a_n|} = \frac{5n+6}{7n+20}$$

so $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{5n+6}{7n+20} = \frac{5}{7} < 1$

$\Rightarrow$ series converges absolutely.

Power Series

<u>Def'n</u> Let x be a variable. A <u>power series</u> is an expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

starting @ 0 now.

where the c_n's are real numbers (called the <u>coefficients</u> of the power series)

<u>ex</u>: ① if $c_n = 1$ for every n we get the power series:

$$\sum_{0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

② if $c_n = 2^n$ for every n:

$$\sum_{0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

↳ can think of a power series as an "infinite polynomial"

→ only once we specify x does a power series become a series in our original sense

→ once we specify an x, can ask about convergence/divergence of the series (for that x)

ex: ① Consider the series from before: $\sum_{0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

if $x = 1/2$, series becomes:

$$\sum_{0}^{\infty} (1/2)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= 1 + \sum_{1}^{\infty} (1/2)^n \leftarrow \text{geo series converging to } 1$$

$$= 1 + 1 = 2$$

but if $x = 1$ series become:

$$\sum_{0}^{\infty} (1)^n = 1 + 1 + 1 + \cdots$$

which diverges.

Notice a power series $\sum_0^\infty c_n x^n$ always converges if $x=0$, since in this case:

$$\sum_0^\infty c_n 0^n = c_0 + 0 + 0 + \cdots = c_0$$

The question: given a power series $\sum_{n=0}^\infty c_n x^n$, for which x's (besides 0) does the series converge?

↳ ratio and root tests can often help to answer!

ex: for our series:

$$\sum_0^\infty x^n = 1 + x + x^2 + \cdots$$

if we imagine plugging in for x, so this becomes an actual series, we have $a_n = x^n$, so:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} |x| = |x|$$

so: if $|x| < 1$, i.e. $-1 < x < 1$,
series converges by ratio test

if $|x| > 1$, i.e. $x > 1$ or $x < -1$,
series diverges by ratio test.

if $x = 1$, series is:
$\sum_0^\infty 1^n = 1 + 1 + 1 + \cdots$ which diverges

if $x = -1$, series is:
$\sum_0^\infty (-1)^n = 1 - 1 + 1 - 1 + \cdots$ which diverges

we've shown: $\sum_0^\infty x^n$ converges if
and only if $-1 < x < 1$.

ex: for which x does the
series $\sum_{n=0}^\infty \frac{1}{n!} x^n$ converge?
" $1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$