Sol'n a) $1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{10} = s_{10} = .6456\ldots$ (141)

b) by thm:

$$|R_{10}| \le b_{11} = \frac{1}{11} = .0909\ldots$$

i.e. $-.0909\ldots \le R_{10} \le .0909\ldots$

c) want $|R_n| < .001$

we know $|R_n| \le \frac{1}{n+1}$

so want $\frac{1}{n+1} \le .001 = \frac{1}{1000}$

$\Rightarrow n+1 \ge 1000$

$\Rightarrow n \ge 999.$

Ratio and Root tests

Tests for convergence/divergence so far:

Test	Applies to
· Divergence test	· all series
· Geometric series test	· geometric series $\sum_1^\infty ar^{n-1}$
· p-series test	· p-series $\sum_1^\infty \frac{1}{n^p}$
· Integral test	· series $\sum_1^\infty a_n = \sum_1^\infty f(n)$ where $f(x) \geq 0$ and decreasing on $[1,\infty)$
· Comparison tests	· series $\sum_1^\infty a_n$ w/ $a_n \geq 0$
· limit comparison test	· series $\sum_1^\infty a_n$ w/ $a_n \geq 0$
· Alt. series test	· alternating series $\sum_1^\infty (-1)^{n-1} b_n$

- today: ratio and root tests
- both powerful - based on geo. series test.

Defn - A series $\sum_1^\infty a_n$ is absolutely convergent if $\sum_1^\infty |a_n|$ converges.

- A series $\sum_1^\infty a_n$ is conditionally convergent if $\sum_1^\infty a_n$ converges but $\sum_1^\infty |a_n|$ diverges.

Ex's - $\sum_1^\infty (-1)^{n-1}\left(\frac{1}{n^2}\right) = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$ is absolutely convergent since $\left|(-1)^{n-1}\frac{1}{n^2}\right| = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.

- $\sum_1^\infty (-1)^{n-1}\left(\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent since it converges but $\sum_1^\infty \frac{1}{n}$ doesn't.

- $\sum_1^\infty (-1)^{n-1}\frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$ diverges by divergence test (neither abs. nor cond. convergent).

Theorem: if $\sum a_n$ is abs. convergent then it is convergent.

Pf: - Sps $\sum_1^\infty |a_n|$ converges.

- observe: $0 \le a_n + |a_n| \le 2|a_n|$ for every n

- since $\sum |a_n|$ converges, we know $\sum 2|a_n| = 2\sum |a_n|$ converges too.

- by comparison: $\sum a_n + |a_n|$ converges

- we can apply our summation rules: since $\sum |a_n|$, $\sum a_n + |a_n|$ conv. then:

$$\sum a_n + |a_n| - \sum |a_n| = \sum a_n \text{ converges.} \checkmark$$

Ex: Show $\sum_1^\infty \frac{\sin(n^2+e^n)}{n^3}$ converges.

Sol'n: - series has both + and − terms, but isn't alternating.

- but notice:

$$\left|\frac{\sin(n^2+e^n)}{n^3}\right| \le \frac{1}{n^3} \quad \text{for all } n$$

- so by comparison:

$$\sum_{1}^{\infty} \left| \frac{\sin(n^2+e^n)}{n^3} \right| \quad \text{converges}$$

- by thm: $\sum_{1}^{\infty} \frac{\sin(n^2+e^n)}{n^3}$ converges. ✓

Note: converse to thm not true of course: e.g. $\sum (-1)^{n-1} \frac{1}{n}$ converges but not abs. conv. (ergo "conditionally" convergent)

Theorem (Ratio test) Sps $\sum_{1}^{\infty} a_n$ is a series

(i) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum a_n$ is absolutely convergent (hence convergent)

(ii) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then $\sum a_n$ diverges (includes if $L = \infty$)

(iii) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ or DNE, test is inconclusive.

Pf (sketch): Sps $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$.

We'll prove $\sum |a_n|$ converges.

↳ there is an integer N, and real number $L < r < 1$ such that:

for every $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| \leq r$.

i.e. $|a_{n+1}| \leq r|a_n|$

↳ so: $|a_{N+1}| \leq r|a_N|$

and: $|a_{N+2}| \leq r|a_{N+1}| \leq r^2|a_N|$

and: $|a_{N+3}| \leq r^3|a_N|$ etc.

→ in general: $|a_{N+k}| \leq r^k|a_N|$

So if we consider the tail sum $\sum_{n=N}^{\infty} |a_n|$ beginning @ N, by comparison we have:

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^{n-1}$$

← geo series w/ $r < 1$ conv. to $\frac{|a_N|}{1-r}$

Hence the tail sum $\sum_{n=N}^{\infty} |a_n|$ converges

Hence the entire sum $\sum_{n=1}^{\infty} |a_n|$ converges.

$$= |a_1| + \cdots + |a_{N-1}| + \sum_{N}^{\infty} |a_n|$$

(ii) same idea, but instead bound $|a_{N+k}|$ below by $r^k |a_N|$ with $r > 1$. Apply divergence test.

(iii) if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, can't use geo series to compare, so test inconclusive.

Ex's: (1) Does $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converge absolutely?

Sol'n: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} (n+1)^3 / 3^{n+1}}{(-1)^n \; n^3 / 3^n} \right|$

$$= \left| (-1) \frac{(n+1)^3 \cdot 3^n}{n^3 \quad 3^{n+1}} \right|$$