# Estimating series using comparison

We showed: Given a series

$$\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} f(n) \leftarrow \text{positive decr.}$$

then:

$$\int_{n+1}^{\infty} f(x) \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

$$= \sum_{1}^{\infty} a_n - S_n = a_{n+1} + a_{n+2} + \dots$$

even if we can't evaluate $\int f(x)\,dx$ easily, can compare to an easier integral to still estimate $R_n$.

ex: a) estimate $\displaystyle\sum_{1}^{\infty} \frac{1}{n^5+5}$ using $S_{10}$.

b) Bound error in this estimate using a comparison

Sol'n: a) $S_{10} = \dfrac{1}{1^5+5} + \dfrac{1}{2^5+5} + \dots + \dfrac{1}{10^5+5}$

$\approx .19926\dots$

b) From above:

$$R_{10} = \sum_{1}^{\infty} \frac{1}{n^5+5} - S_{10} \leq \int_{10}^{\infty} \frac{1}{x^5+5}\, dx$$

hard. would need to ~~[illegible]~~ factor, do p.f.d.

but $\frac{1}{x^5+5} \leq \frac{1}{x^5}$ for $x \geq 0$

So $\int_{10}^{\infty} \frac{1}{x^5+5}\, dx \leq \int_{10}^{\infty} \frac{1}{x^5}\, dx$ ← easier

$$= \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x^5}$$

$$= \lim_{t\to\infty} -\frac{1}{4x^4}\Big|_{10}^{t}$$

$$= 0 - \left(-\frac{1}{40,000}\right)$$

$$= \frac{1}{40,000} = .000025$$

so: $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^5+5}\, dx \leq \int_{10}^{\infty} \frac{1}{x^5}\, dx = .000025$

$\Rightarrow$ $S_{10} = .19926$ is w/in .000025 of entire series $\sum_{1}^{\infty} \frac{1}{n^5+5}$

## Alternating Series



**Def'n** an alternating series is a series of one of the two forms:

$$\sum_{1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

$$\sum_{1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

where $b_n \geq 0$ for all $n$.

e.g.

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-2 + 4 - 8 + 16 - \cdots = \sum_{1}^{\infty} (-1)^n 2^n$$

are alternating but

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots$$

is not (needs to alternate sign every other term)

Q: when do alternating series converge?

Recall: (divergence test): convergence of $\sum_{n=1}^{\infty} a_n$ implies $\lim_{n\to\infty} a_n = 0$.

- converse not true: $\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{1}^{\infty} a_n$ converges in general (e.g $\sum \frac{1}{n}$)

- but for alternating series, converse is true, as long as terms are decreasing in abs. value:

**Theorem** Given an alternating series $\sum_{1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \cdots$ $(b_n \geq 0)$

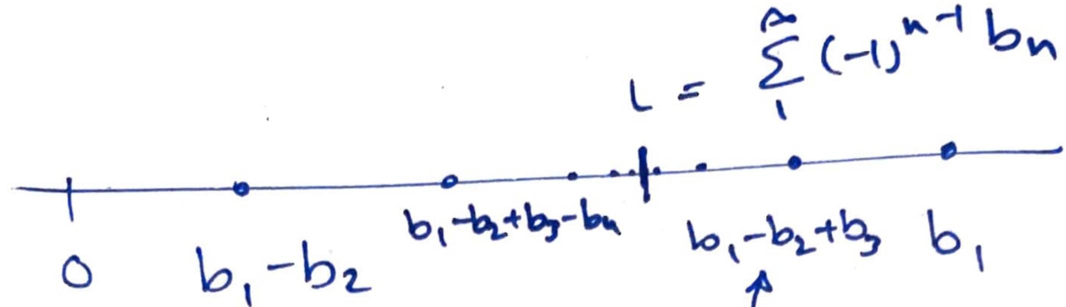
If: ① $b_n \geq b_{n+1}$ for all $n$

② $\lim_{n\to\infty} b_n = 0$

Then: $\sum_{1}^{\infty} (-1)^{n-1} b_n$ converges

(same true for $\sum_{1}^{\infty} (-1)^n b_n$ form)

## Proof by picture




doesn't go below 0 since $b_2 \le b_1$

doesn't go above $b_1$ since $b_3 \le b_2$

etc.

Since $\lim_{n\to\infty} b_n = 0$, jumps get arbitrarily small, hence limit of partial sums exists $= \sum (-1)^{n-1} b_n$.

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**ex's** ① the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

**why**: ① $\frac{1}{n+1} \le \frac{1}{n}$ for all $n$

② $\lim_{n\to\infty} \frac{1}{n} = 0$

Theorem says: series converges.

(7) The series $\sum_1^\infty (-1)^{n-1} \frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \dots$

is alternating, but does not converge.

$\lim_{n\to\infty} \frac{n}{n+1} \neq 0$ hence theorem doesn't apply. In fact $\lim_{n\to\infty} \frac{n}{n+1} = 1$ so $\lim_{n\to\infty} (-1)^n \frac{n}{n+1}$ DNE. By divergence test, series diverges.

(8) What about $\sum_1^\infty (-1)^{n+1} \frac{n}{e^n}$ ?

Soln: Looks like $\frac{n}{e^n}$ is decreasing — but is it? Yes.

Observe: $\frac{d}{dx} \frac{x}{e^x} = \frac{e^x - xe^x}{(e^x)^2}$

$$= \frac{1-x}{e^x}$$

If $x \geq 1$, $1-x/e^x \leq 0$

So $x/e^x$ is decreasing on $[1,\infty)$

so $n/e^n$ is too, i.e.

$$\frac{n+1}{e^{n+1}} \leq \frac{n}{e^n} \quad \text{for all } n \quad (i)$$

Furthermore:

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0 \quad ②$$

$① + ② \Rightarrow \sum_1^\infty (-1)^{n+1} \frac{n}{e^n}$ converges, by thm.

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## Estimates for alternating series

- We've learned how to estimate series with integrals ... but integrals are hard!
- turns out: For alt. series $\sum_1^\infty (-1)^{n-1} b_n$ with decreasing terms, the error $R_n = \sum_1^\infty (-1)^{n-1} b_n - S_n$ is easy to estimate.
- no integrals required!

**Theorem**: If $\sum_{1}^{\infty} (-1)^{n-1} b_n$ is an alt. series such that:

so series converges

① $b_{n+1} \le b_n$ for all $n$

② $\lim_{n\to\infty} b_n = 0$

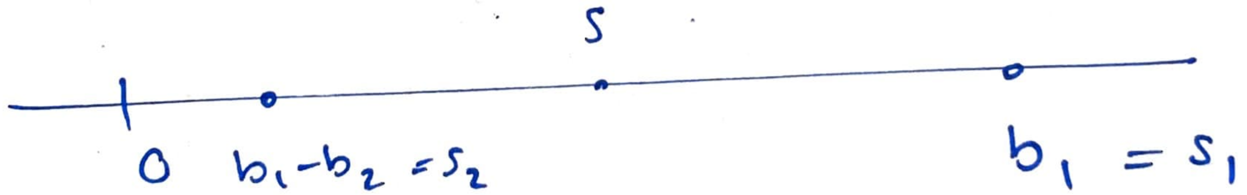
**Then**: $|R_n| \le b_{n+1}$

i.e. $-b_{n+1} \le R_n \le b_{n+1}$

Proof by (the same) picture:

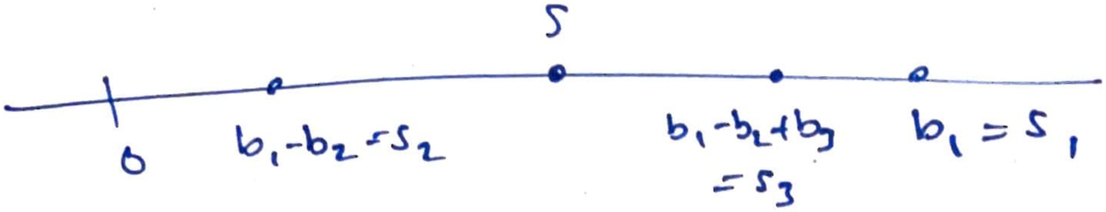
Let $s = \sum_{1}^{\infty} (-1)^{n-1} b_n$. Then $|R_n| = |s - s_n|$

= the positive distance from $s$ to $s_n$




distance from $s$ to $s_1$ $\le$
distance from $s_2$ to $s_1$

i.e. $|R_1| \le b_2$




distance from $S$ to $S_2 \le$
distance from $S_2$ to $S_3$

i.e. $|R_2| \le b_3$

... and so on!

in general: $S$ between $S_n$ and $S_{n+1}$
so that: distance from $S$ to $S_n$
$\le$ distance from $S_{n+1}$ to $S_n$

i.e. $|R_n| \le |S_{n+1} - S_n| = b_n$ ✓

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<u>Ex</u>: Consider $\sum_{1}^{\infty} (-1)^{n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

a) use $S_{10}$ to estimate series

b) bound the error $R_{10}$ in this estimate

c) How large must $n$ be to guarantee $|R_n| \le .001$?