Sol'n a) $S_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \dots + \frac{1}{1000}$.

$= 1.1975$

by theorem:

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

not hard to check:

$$\int_{K}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2K^2}$$

So we get:

$$\frac{1}{2 \cdot 11^2} \le R_{10} \le \frac{1}{2 \cdot 10^2}$$

$$\Rightarrow \underbrace{\frac{1}{242} \le R_{10} \le \frac{1}{200}}$$

tells us the "error":

$R_{10} \le \frac{1}{200} = .005$

So: the actual sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$= S_{10} + R_{10}$

$= 1.1975 + R_{10}$

$\Rightarrow \sum_{1}^{\infty} \frac{1}{n^3}$ is w/in $.005$ of 1.1975.

Looking at both sides of inequality gives more information:

Can rewrite:

$$\frac{1}{242} \le R_{10} \le \frac{1}{200}$$

$$\Rightarrow \frac{1}{242} \le \left(\sum_1^\infty \frac{1}{n^3}\right) - S_{10} \le \frac{1}{200}$$

$$\Rightarrow S_{10} + \frac{1}{242} \le \sum_1^\infty \frac{1}{n^3} \le S_{10} + \frac{1}{200}$$

$$S_{10} + \frac{1}{242} = 1.1975 + \frac{1}{242} = 1.2016\ldots \qquad S_{10} + \frac{1}{200} = 1.1975 + \frac{1}{200} = 1.2025$$

$$1.2016\ldots \le \sum_1^\infty \frac{1}{n^3} \le 1.2025$$

diff only .00086...

b) want: $R_n < .0005$

know $R_n \le \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$

set: $\frac{1}{2n^2} < .0005$

$\Rightarrow n^2 > 1000 \Rightarrow n^2 > 31.62$

So $n = 32$ guarantees it.

11.4 The comparison tests

Going forward, you may assume the following:

Thm ("p-test")

(1) $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p>1$

Pf: ez to check.

Hence, by integral test we have:

(2) $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only $p>1$.

Back to series:

ex: does $\sum^\infty \frac{1}{2^n+n}$ converge?

- integral test fails us: $\int \frac{1}{2^x+x}\,dx$ not easy to evaluate.

- but observe:

$$\frac{1}{2^n+n} < \frac{1}{2^n} \quad \text{for every } n.$$

- so we have:

$$\sum_{n=1}^\infty \frac{1}{2^n+n} < \sum_{n=1}^\infty \frac{1}{2^n} = 1$$

in particular $\uparrow$ converges.

ex: what about $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$?

Sol'n: - already checked w/ integral test series diverges; p-test works too.

- another way: observe $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for every n.

So: $\sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

diverges to ∞ so diverges too.

More generally we have:

<u>Theorem</u> ("Comparison tests")

Suppose $\sum_1^{\infty} a_n$, $\sum_1^{\infty} b_n$ are series w/ positive terms.

① if $a_n \leq b_n$ for every n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_1^{\infty} a_n$ converges.

② if $a_n \geq b_n$ for every n and $\sum_1^{\infty} b_n$ diverges then $\sum_1^{\infty} a_n$ diverges.

- these tests are useful (see ex's above) but don't always get us there:

<u>ex</u>: Does $\sum_1^\infty \frac{1}{2^n - n}$ converge?

instinct: <u>yes</u> since "looks like" $\sum_1^\infty \frac{1}{2^n}$ but direct comparison fails since

$$\frac{1}{2^n} < \frac{1}{2^n - n} \quad \text{for every } n.$$

So convergence of $\sum_1^\infty \frac{1}{2^n}$ doesn't imply convergence of $\sum_1^\infty \frac{1}{2^n - n}$.

<u>Theorem</u> (Limit comparison test)

Suppose $\sum_1^\infty a_n$, $\sum_1^\infty b_n$ are series w/ positive terms, <u>and</u>

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } 0 < c < \infty$$

<u>then</u>: $\sum_1^\infty a_n$ converges if and only if $\sum_1^\infty b_n$ converges

pf: there is N s.t. for all $n > N$

$$\frac{c}{2} \leq \frac{a_n}{b_n} \leq 2c$$

$$\Rightarrow \frac{c}{2} b_n \leq a_n \leq 2c\, b_n$$

$$\Rightarrow \frac{c}{2} \sum b_n \leq \sum a_n \leq 2c \sum b_n$$

if $\sum b_n$ div then $\sum a_n$ div

if $\sum b_n$ conv. then $\sum a_n$ conv.

back to our ex:

Observe: $\lim_{n\to\infty} \frac{1/2^n}{1/(2^n-n)} = \lim_{n\to\infty} \frac{2^n-n}{2^n}$

$= \lim_{n\to\infty} 1-\frac{n}{2^n} = \lim_{x\to\infty} 1-\frac{x}{2^x} \overset{L'H}{=} \lim_{x\to\infty} 1-\frac{1}{(\ln 2)2^x}$

$= 1-0 = 1$

So, since $0<1<\infty$, by limit comparison we have

$\sum_1^\infty \frac{1}{2^n-n}$ converges since $\sum_1^\infty \frac{1}{2^n}$ does.

ex: Does $\sum_{n=1}^\infty \frac{2n^2+3n}{\sqrt{5+n^5}}$ converge?

instinct: $\frac{2n^2+3n}{\sqrt{5+n^5}}$ looks like $\frac{2n^2}{\sqrt{n^5}}$

$= \frac{2n^2}{\sqrt{n}\,n^2}$

$= \frac{2}{\sqrt{n}}$

We know $\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges

let's compare terms:

B1

$$\lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}}$$ divide top + bottom by $n^{5/2}$

$$= \lim_{n\to\infty} \frac{2+\cancel{\frac{3}{n}}}{\sqrt{\cancel{\frac{5}{n^5}}+1}} = \frac{2}{\sqrt{1}} = 2$$

since $0<2<\infty$ and $\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges

so must $\sum_1^\infty \frac{2n^2+3n}{\sqrt{5+n^5}}$ ✓