11.3 The Integral Test

We know:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \sum_{n=1}^{\infty}\frac{1}{2^n} \quad \text{converges (actually} = 1)$$

also:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n} \quad \text{diverges}$$

what about

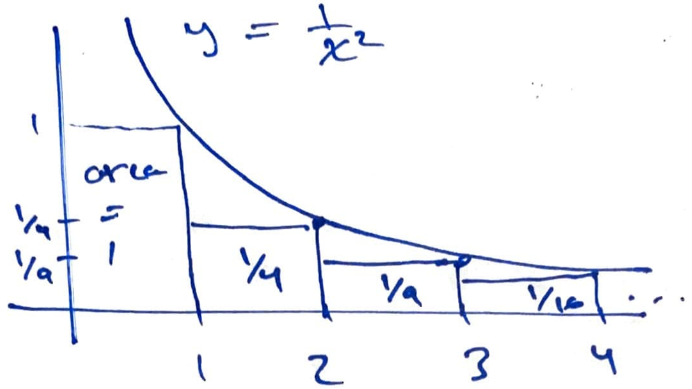
$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} \; ?$$

An issue: there's no nice formula for the partial sum $S_n = 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}$ so, hard to find

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Can we at least determine whether $\sum \frac{1}{n^2}$ converges or diverges?

Can visualize terms in this series as areas of rectangles:



But look: can bound total area of rectangles (except the first) by area under $\frac{1}{x^2}$!

Precisely: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$

$$\leq 1 + \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{\text{we showed } = 1 \text{ earlier}}$$

$$= 1 + 1 = 2.$$

So: using an improper integral we're shown:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2.$$

In particular – series converges!

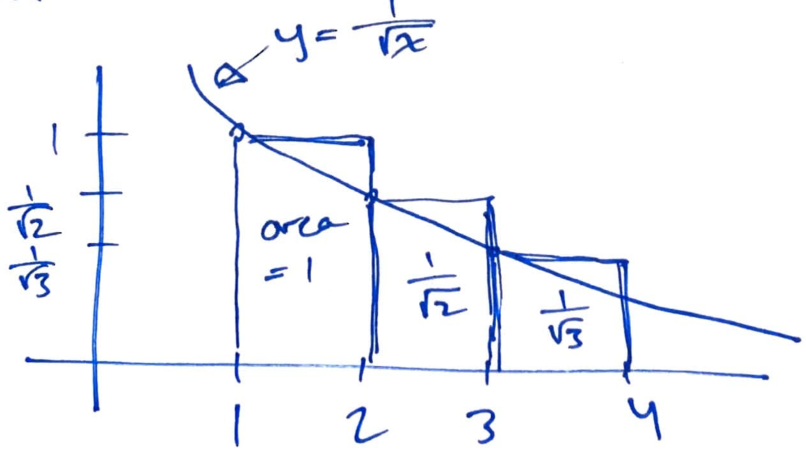
We have not computed exact value

(in fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

Could we also use an integral to show a series diverges?

Consider: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Can visualize:



Observe: difference in how we draw rectangles relative to x-axis.

Hence: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \infty$

sum of rectangles

diverges!

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ must diverge as well!

Observations in these examples are formalized in following:

<u>Theorem</u> (Integral Test) Sps a series is defined by $a_n = f(n)$, where $f(x)$ is a <u>continuous</u>, <u>decreasing</u> function (e.g. $\frac{1}{\sqrt{x}}, \frac{1}{x^2}$) and $f(x) \geq 0$.

Then:

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x)\,dx$$

in particular:

$\sum_{n=1}^{\infty} f(n)$ converges if and only if

$\int_1^{\infty} f(x)\,dx$ converges.

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ex: determine whether the following series converge.

1) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots$

2) $\sum_{n=1}^{\infty} \frac{n^3}{n^4+4} = \frac{1}{4} + \frac{8}{20} + \cdots$

Sol'n: 1) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{1}{x\sqrt{x}}$

$= \frac{1}{x^{3/2}}$

observe: $\frac{1}{x^{3/2}}$ is cts, decreasing, nonnegative

Also: $\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \lim_{t\to\infty} \int_1^t x^{-3/2}\,dx$

$= \lim_{t\to\infty} -2x^{-1/2}\Big|_1^t$

$= \lim_{t\to\infty} \left(-\frac{2}{\sqrt{t}} + 2\right)$

$= 2$

So $\int_1^\infty x^{-3/2}\,dx$ converges

By integral test: $\sum_{n=1}^{\infty} n^{-3/2} = \sum_1^\infty \frac{1}{n\sqrt{n}}$

converges too!

2) $\sum_{n=1}^{\infty} \frac{n^3}{n^4+4} = \sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{x^3}{x^4+4}$

we have: $\int_1^\infty \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^3}{x^4+4}\,dx$

$u = x^4+4$

$du = 4x^3$

$\int \frac{1}{4} u^{-1}\,du$

$= \frac{1}{4}\ln|u|$

$= \frac{1}{4}\ln|x^4+4|$

$= \lim_{t\to\infty} \frac{1}{4}\ln|x^4+4|\Big|_1^\infty$

$= \lim_{t\to\infty}\left(\frac{1}{4}\ln(t^4+4) - \frac{1}{4}\ln(5)\right)$

$= \infty$

so integral diverges

$\Rightarrow \sum_1^\infty \frac{n^3}{n^4+4}$ diverges too!

Estimating $\sum_{n=1}^{\infty} a_n$

Given a series of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ on cts decreasing function

that we can't evaluate exactly (e.g $\sum_{1}^{\infty} \frac{1}{n^2}$), we'd like to be able to estimate it well.

↳ the nth partial sum $S_n = a_1 + a_2 + \cdots + a_n$
$= f(1) + f(2) + \cdots + f(n)$

gives such an estimate.

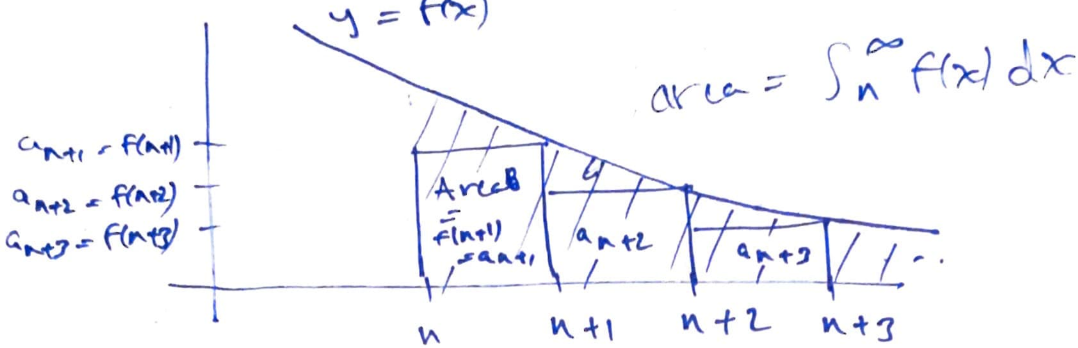
- the larger the n, the better S_n approximates $\sum_{n=1}^{\infty} a_n$
- can we bound the error?

Define: the nth remainder

$$R_n = \sum_{n=1}^{\infty} a_n - S_n$$

$$= a_{n+1} + a_{n+2} + \cdots$$

can estimate R_n with an integral:

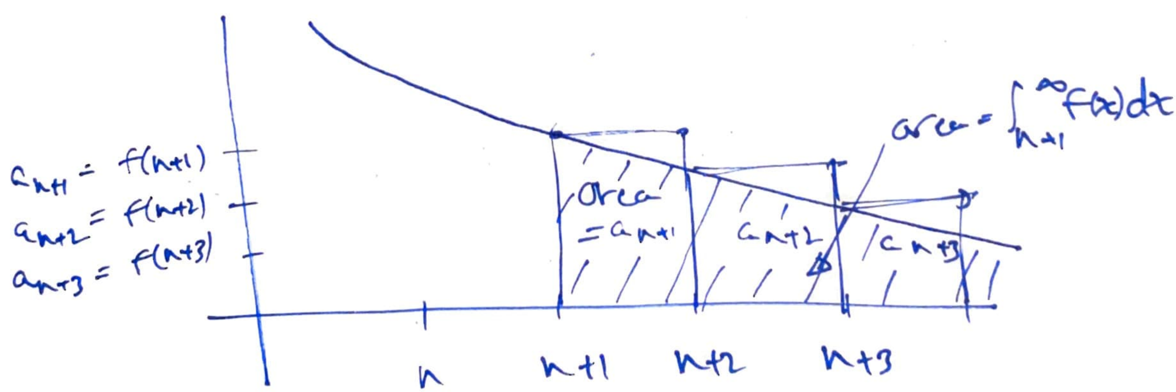


sum of rectangles' areas $= a_{n+1} + a_{n+2} + \cdots$

$$= R_n$$

$$\leq \int_n^\infty f(x)\,dx$$

On the other hand:



sum of rectangles' areas $= a_{n+1} + a_{n+2} + \cdots$

$= R_n$ again

$\geq \int_{n+1}^{\infty} f(x)\,dx$

We've shown:

<u>Theorem</u> if $f(x)$ is cts, decreasing, nonnegative and $R_n = \sum_{n=1}^{\infty} f(n) - S_n$

$= a_{n+1} + a_{n+2} + \cdots$

<u>Then</u>: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$ (Δ)

assuming the series $\sum_{1}^{\infty} f(n)$ converges.

another way of writing this inequality:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq S_n + \int_{n}^{\infty} f(x)\,dx \quad (*)$$

<u>ex</u>: Consider $\sum_{n=1}^{\infty} \frac{1}{n^3}$

a) Compute S_{10}, estimate R_{10}

b) How large does n need to be to guarantee $R_n < .0005$?