Ex's: 1) $\sum_{n=1}^{\infty} 5(1/3)^{n-1} = 5 + 5/3 + 5/9 + \cdots$

$$\overset{\text{Thm}}{=} \frac{5}{1-1/3} = 15/2$$

2) $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

3) $\sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1} = 3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$

$$= \frac{3}{1-(-1/2)} = \frac{3}{\frac{3}{2}} = 2$$

4) $\sum_{n=1}^{\infty} 3\cdot 2^{n}$ diverges $(2\geq 1)$

ex: every real number with an eventually repeating decimal expansion can be written as a fraction!

e.g. consider:

$$1.2343434\ldots = 1.2\overline{34}$$

$$= 1.2 + .034 + .00034 + .0000034 + \cdots$$

$$= 1.2 + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \cdots$$

$$= 1.2 + \frac{34}{10^3} + \frac{34}{10^3}\left(\frac{1}{10^2}\right) + \frac{34}{10^3}\left(\frac{1}{10^2}\right)^2 + \cdots$$

$$= 1.2 + \sum_{n=1}^{\infty} \frac{34}{10^3}\left(\frac{1}{10^2}\right)^{n-1}$$

$$= 1.2 + \frac{\frac{34}{10^3}}{\underbrace{1-\frac{1}{10^2}}_{\frac{99}{100}}} = 1.2 + \frac{34}{1000}\cdot\frac{100}{99}$$

$$= \frac{12}{10} + \frac{34}{990}$$

$$= \frac{1222}{990} \checkmark$$

$$= 1.2343434\ldots$$

<u>ex</u>: (telescoping series)

- not all series are geometric (of course)
- sometimes can show convergence w/ a clever trick.

Consider: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

not geometric – does it converge?

<u>Observe</u>: $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$

So our nth partial sum:

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots}$$

$$\cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$

So we've shown: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

<u>ex</u>: (the harmonic series diverges)

Consider $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

not geometric: does it converge?

Consider the partial sums:

$S_2, S_4, S_8, S_{16}, \cdots, S_{2^n}, \cdots$

$S_2 = 1+\frac{1}{2} = \frac{3}{2} = 1.5$

$S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)$

$= 1+\frac{1}{2}+\frac{1}{2} = 2$

$S_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$

$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 2.5$

Same idea shows in general:

$S_{2^n} = 1+\frac{n}{2}$

hence $\lim_{n\to\infty} S_n = \infty$

So: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

Theorem (the divergence test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Pf: "clear" (see book)

the point: if $\lim_{n\to\infty} a_n \neq 0$ or DNE

then $\sum_{n=1}^{\infty} a_n$ must diverge!

ex: the series

$$\sum_{n=1}^{\infty} \frac{n^3+1}{n(n^2+1)}$$

diverges.

Why: $$\lim_{n\to\infty} \frac{n^3+1}{n(n^2+1)} = \lim_{n\to\infty} \frac{n^3+1}{n^3+n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^3}}{1+\frac{1}{n^2}}$$

$$= 1 \neq 0$$

Summation rules

Theorem if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and c is a constant, then:

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

and $$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

ex: $$\sum_{n=1}^{\infty} 5\left(\tfrac{1}{2}\right)^{n-1} - 3\left(\tfrac{1}{7}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 5\left(\tfrac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} 3\left(\tfrac{1}{7}\right)^{n-1}$$

$$= \frac{5}{1-\frac{1}{2}} - \frac{3}{1-\frac{1}{7}}$$

$$= 10 - \frac{7}{2}$$

$$= \frac{13}{2}$$