A semantic criterion for elim'n of quantifiers

- inspired by our approach to DLO we formulate two algebraic conditions that together imply quant. elim'n

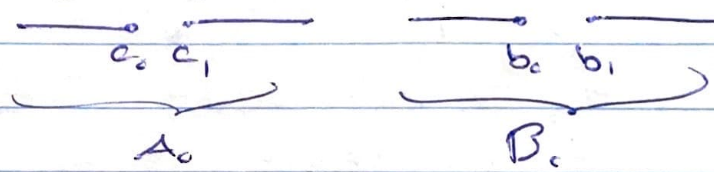
<u>Def'n</u> A theory T has the <u>isomorphism condition</u> if, given models $\mathcal{A}, \mathcal{B} \models T$ with two isomorphic substructures (not necessarily models of T) $\mathcal{A}_0, \mathcal{B}_0$ along w/ an isomorphism $\pi_0 : \mathcal{A}_0 \to \mathcal{B}_0$ there exist substructures $\mathcal{A}_1, \mathcal{B}_1 \models T$ w/ $\mathcal{A}_0 \leq \mathcal{A}_1 \leq \mathcal{A}$, $\mathcal{B}_0 \leq \mathcal{B}_1 \leq \mathcal{B}$ and an isomorphism $\pi_1 : \mathcal{A}_1 \to \mathcal{B}_1$ extending π_0.

- Let's motivate def'n by checking DLO has iso condition.

- Sps $\mathcal{A}, \mathcal{B} \models DLO$ and $\mathcal{A}_0 \leq \mathcal{A}$, $\mathcal{B}_0 \leq \mathcal{B}$ are suborders w/ isomorphism $\pi_0 : \mathcal{A}_0 \to \mathcal{B}_0$
- if $\mathcal{A}_0, \mathcal{B}_0 \models DLO$, just take $\mathcal{A}_1 = \mathcal{A}_0$ and $\mathcal{B}_1 = \mathcal{B}_0$

- If $A_0, B_0 \not\models$ DLO then e.g. A_0 has a jump (⟵ A_0) or a top pt or a bottom point; and the same for B_0.

- If $a_0 < a_1$ is a jump in A_0 (no pts between) then $b_0 = \pi_0(a_0) < \pi_0(a_1) = b_1$ is a jump in B_0



- By density of A can find $a_{1/2} \in A$ s.t. $a_0 < a_{1/2} < a_1$; likewise $b_0 < b_{1/2} < b_1$ in B.

- likewise can find $a_{1/4}, a_{3/4}$ and $b_{1/4}, b_{3/4}$

- etc: eventually get a dense collection of pts a_i (i dyadic) in A and b_i in B and can extend π_0 by letting $a_i \to b_i$

- If we do this for every jump and also handle any top/bottom

points (using that $\mathcal{A}, \mathcal{B}$ have none)
eventually get $\mathcal{A}_1, \mathcal{B}_1 \models DLO$
extending $\mathcal{A}_0, \mathcal{B}_0$ and on wo
$\pi_1 : \mathcal{A}_1 \to \mathcal{B}_1$ extending π_0. ✓

Here's the second condition.

<u>Def'n</u> a theory T satisfies the <u>submodel condition</u> if for two structures $\mathcal{A} \leq \mathcal{B}$ both models of T, and any simple formula $\varphi(\vec{x}) := \exists y (\chi_1(\vec{x}) \wedge \dots \wedge \chi_n(\vec{x}))$ with k free variables, and $\vec{c} \in A^k$ we have:

$$\mathcal{A} \models \varphi(\vec{c}) \iff \mathcal{B} \models \varphi(\vec{c})$$

Intuition: says that if a statement
"$\exists y$ (q.f. part)" is true in $\mathcal{B}$
(i.e. there is $b \in B$ making q.f. part true)
then actually
"$\exists y$ (q.f. part)" is already
true in $\mathcal{A}$
(i.e. ~~can~~ actually find the witness $b \in A$)

q.f. part = parameters, from $\mathcal{A}$.

- DLO has submodel condition:
such $\varphi(\vec{c})$ assert "$\exists y$ that is above

and below the fixed points $c_1, \ldots, c_k$ etc"
by density there is such a y in B if and only if there is one in A. ✓

We want to show:

<u>Theorem</u> If a theory T satisfies the isomorphism and submodel conditions, then T has quant. elim'n.

We'll need the following corollary of the completeness theorem:

<u>Lemma</u>: Sps T is a theory in a language L and σ is a sentence in L.

<u>TFAE</u>: ① $T \vdash \sigma \Leftrightarrow \sigma^*$ for some q.f. σ^*
② Whenever $A, B \models T$ and satisfy the same q.f. sentences, then $A \models \sigma$ iff $B \models \sigma$.

Pf: ($\Rightarrow$) immediate
($\Leftarrow$) - As warmup, consider case when L has no constants
- then any pair $A, B \models T$ satisfy same q.f. sentences - there are no q.f. sentences.

- So ② becomes: if $\mathcal{A}, \mathcal{B} \models T$ then $\mathcal{A} \models \sigma$ iff $\mathcal{B} \models \sigma$, i.e. σ is true in every model, or false in every model.
- In first case, by completeness theorem, $T \vdash \sigma$, i.e. $T \vdash \sigma \Leftrightarrow \top_{q.f}$
- in second $T \vdash \neg\sigma$ i.e. $T \vdash \sigma \Leftrightarrow \bot_{q.f.}$

<u>Now in general:</u>

- let S be set of q.f. sentences φ s.t. $T \cup \{\sigma\} \vdash \varphi$.
- We claim $T \cup S \vdash \sigma$

↳ if can prove, done: since by finiteness of proofs would then have $T \cup \{\varphi_1, .., \varphi_n\} \vdash \sigma$ for some finite # of q.f. φ_i's.

↳ Thus $T \vdash \varphi_1 \wedge .. \wedge \varphi_n \Rightarrow \sigma$ and by assumption on S $T \vdash \Leftarrow$ direction already so $T \vdash \sigma \Leftrightarrow \underbrace{\varphi_1 \wedge .. \wedge \varphi_n}_{q.f.}$

So we prove the claim

- if claim fails by completeness there is a model $\mathcal{A} \models T \cup S$ s.t. $\mathcal{A} \models \neg\sigma$.

- let S' = set of q.f. sentences true in $\mathcal{A}$ (so for each q.f sent. σ, exactly one of $\sigma, \neg\sigma \in S'$)
- Observe $S' \supseteq S$
- if $\mathcal{B} \models T \cup S'$, then $\mathcal{A}, \mathcal{B}$ model same q.f. sentences and also T
- Follows: $\mathcal{B} \models \neg\sigma$ by hypothesis ②

↳ since $\mathcal{B}$ arbitrary by completeness theorem $T \cup S' \vdash \neg\sigma$ ← q.f.

- as above there is actually a single sentence $\sigma' \in S'$ (viewed as a conjunction needed for proof) s.t. $T \cup \{\sigma'\} \vdash \neg\sigma$
- i.e. $T \vdash \sigma' \Rightarrow \neg\sigma$

~~[illegible]~~ i.e. $T \vdash \sigma \Rightarrow \neg\sigma'$

i.e. $T \cup \{\sigma\} \vdash \neg\sigma'$
- but then $\neg\sigma' \in S$, contradicting $\sigma \in S$. ✓

We can now prove theorem above.

Pf: - Sps T satisfies isomorphism and submodel conditions
- to show T has quant. elim'n

sufficient to check for simple formulas.

- Sps $\varphi(x_1, .., x_m) := \exists y(\chi_1(\vec{x}, y) .. \chi_k(\vec{x}))$ is simple.
- We extend lang L to L' by adding new constants $c_1, .., c_m$ (one for each var. x_i in φ)
- may now view T as a theory in L'
- We will find a q.f. $\underline{\underline{L'}}$-sentence σ^* s.t.

$$T \vdash \varphi(c_1, .., c_m) \Leftrightarrow \sigma^*$$

↳ if we can do this: will be done

<u>Why</u>: Since T has no axioms concerning the c_i we then have $T \vdash \varphi(\vec{x}) \Leftrightarrow \varphi^*(\vec{x})$ where φ^* is unique L-formula s.t. $\sigma^* = \varphi^*(\vec{c})$

(intuitively plausible but can be verified by looking @ our deduction scheme: any proof of $\varphi(\vec{c}) \Leftrightarrow \sigma^*$ can be turned into a proof of $\varphi(\vec{x}) \Leftrightarrow \varphi^*(\vec{x})$)

(or, semantically: if not, there is model of T where $\forall \vec{x}\,(\varphi(\vec{x}) \Leftrightarrow \varphi^*(\vec{x}))$ is false.. but we can in this model interpret the c_i's as a witness to this failure, contradiction)

- So we check that lemma applies to $\varphi(c_1, \ldots, c_m)$
- Sps $\mathcal{A}, \mathcal{B}$ are L'-structures w/ $\mathcal{A}, \mathcal{B} \models T$ that satisfy same ~~[illegible]~~ q.f. L'-sentences
- <u>WTS</u>: $\mathcal{A} \models \varphi(\vec{c}) \Leftrightarrow \mathcal{B} \models \varphi(\vec{c})$ i.e. $\varphi(\vec{c})$ is true in both or false in both.

- Consider the substructures
  $A_0 = \{t^{\mathcal{A}} : t \text{ a } L'\text{-term}\}$
  $B_0 = \{t^{\mathcal{B}} : t \text{ a } L'\text{-term}\}$ of $\mathcal{A}, \mathcal{B}$ resp.
- these are the minimal substructures
- the map $t^{\mathcal{A}} \to t^{\mathcal{B}}$ is an isomorphism between A_0, B_0

  <u>Why</u>: if not an isomorphism there is a q.f. sentence (indeed an atomic one!) that witnesses this contradicting $\mathcal{A}, \mathcal{B}$ satisfy same q.f. sentences

- if now we view $\mathcal{A}, \mathcal{B}$ as L-structures, map above is still an isomorphism between A_0, B_0
- by iso condition A_0, B_0 may be extended to ~~[illegible]~~ $\mathcal{A}', \mathcal{B}' \models T$ within $\mathcal{A}, \mathcal{B}$

- But then $A_1 \models \varphi(c_1, \ldots, c_m) \Leftrightarrow$
  $B_1 \models \varphi(c_1, \ldots, c_m)$
  since A_1, B_1 isomorphic (so satisfy same formulas)
  (here we are viewing c_i as their interpretations c_i^{A} and c_i^{B})

- but then by submodel condition we have $A \models \varphi(\vec{c}) \Leftrightarrow A_1 \models \varphi(\vec{c}) \Leftrightarrow B_1 \models \varphi(\vec{c}) \Leftrightarrow B \models \varphi(\vec{c})$

  ↳ i.e. A, B agree on $\varphi(\vec{c})$
  ↳ by lemma $T \vdash \varphi(\vec{c}) \Leftrightarrow \sigma^*$ for some q.f. σ^*. ✓

Algebraically closed fields

- We'll apply theorem to show theory of algebraically closed fields (ACF) has quant elim
- ACF consists of following axioms (lang is $L = \{+, \cdot, 0, 1\}$)
  - axioms for fields
  - for every $n \geq 1$ the axiom

$$\forall x_0 \ldots \forall x_n \left(x_n \neq 0 \Rightarrow \exists y \left(x_n y^n + \ldots + x_1 y + x_0 = 0\right)\right)$$