Decidable Theories

- A countable language $L = \{R_i\} \cup \{f_j\} \cup \{c_k\}$ is called recursive if the set of tuples $\{(n, i, m, j, k) : n$ is arity of R_i, m is arity of $f_j\}$ is recursive (as a subset of $\mathbb{N}^5$)

- All finite langs are trivially recursive; we may now consider ctbly infinite langs that are recursive.

- For L recursive, can compute whether a sequence of symbols is a fmla, talk about recursive sets of fmlas, etc.

- Recall: a theory T of L-formulas is called decidable if $\text{Conseq}(T)$ is recursive.

- We've seen several examples of natural theories that are undecidable (e.g. theory of groups, of rings, of graphs, of posets...)
  - but there are also natural examples of decidable theories!

Our main tool for proving decidability results will be <u>quantifier elimination</u>

<u>Def'n</u> A theory T admits <u>elimination of quantifiers</u> for a formula $\varphi(x_1, .., x_n)$ if there is a quantifier free formula $\varphi^*(x_1, .., x_n)$ s.t.

$$T \vdash \varphi(x_1, .., x_n) \Leftrightarrow \varphi^*(x_1, .., x_n)$$

- e.g. let T be the complete theory of the ordered field of real numbers $(\mathbb{R}, \leq, +, \cdot, -, ^{-1}, 0, 1)$
- Consider $\varphi(a,b,c) := \exists x\,(ax^2 + bx + c = 0)$
- then T proves φ is equiv to fmla

$$\varphi^*(a,b,c) := (a \neq 0 \wedge b^2 - 4ac \geq 0) \checkmark$$
$$(a = 0 \wedge b \neq 0) \checkmark$$
$$(a = b = c = 0)$$

(just run the usual proof of derivation of quadratic formula)

- Observe: if φ has no free vars (i.e. is a sentence) then if our lang L has no constants it's impossible for $T \vdash \varphi \Leftrightarrow \varphi^*$ for some q.f. φ^*

- We want sentences φ s.t. $T \vdash \varphi$ or $T \vdash \neg\varphi$ to admit elim'n of quants

So we introduce formal symbols 0/1 (or T/$\perp$ if preferred) which we use as φ^* for such φ.

- (so eg if $T \vdash \varphi$ we say $T \vdash \varphi \Leftrightarrow \varphi^*$ where $\varphi^* := 1$)

<u>Def'n</u> T admits elim'n of quantifiers if it admits it for every formula φ.

- our first task is to reduce elim'n of quants for general φ to φ of a special form
- call a fmla <u>simple</u> if it has the form $\exists x(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n)$ where each φ_i is either atomic (i.e. $t = s$ or $R_i(t_1, \dots, t_k)$) or negation of atomic.

<u>Prop'n</u> Sps T is a theory. Then T admits elim'n of quantifiers iff it admits it for simple formulas

Pf: need only show ($\Leftarrow$) Induct on construction of a given φ

- if φ atomic then $\varphi^* := \varphi$ is q.f. and $T \vdash \varphi \Leftrightarrow \varphi^*$
- if φ is $\neg\chi$ and $T \vdash \chi \Leftrightarrow \chi^*$ for χ^* q.f. then $T \vdash \varphi \Leftrightarrow \neg\chi^*$
- if φ is $\chi \wedge \sigma$, take $\varphi^* := \chi^* \wedge \sigma^*$

- Finally sps φ is $\exists x \chi$ and $T \vdash \chi \Leftrightarrow \chi^*$ for some q.f. χ^*

Fact Any quantifier free fmla is (provably over any T) equiv. to a fmla in disjunctive normal form i.e. to a fmla $\chi_1 \vee \chi_2 \vee .. \vee \chi_n$ where each χ_i is a conjunction of atomic/neg'n of atomic fmlas

Pf: exercise.

- Now: $T \vdash \chi \Leftrightarrow \chi^*$ and by Fact $\Leftrightarrow \chi_1 \vee \dots \vee \chi_n$ for some q.f. χ_i.

- But then $T \vdash \underbrace{\exists x \chi}_{\varphi} \Leftrightarrow \exists x \chi_1 \vee \exists x \chi_2 \vee \dots \exists x \chi_n$
  ($\exists$'s distrib over $\vee$'s)
- but by induction $T \vdash \exists x \chi_i \Leftrightarrow \chi_i'$ for some q.f. χ_i', $\forall i$,
  $\Rightarrow T \vdash \varphi \Leftrightarrow \chi_1' \vee \dots \vee \chi_n'$ which is q.f. ✓

Some decidable theories

Dense linear orders:

- Sps $L = \{<\}$ is a lang w/ a single binary rel'n symbol $<$
- DLO denotes the theory consisting of following axioms:
  1. "$<$ is a linear order"
     (i.e. $\forall x, y, z \, (x \not< x \wedge (x<y \wedge y<z \Rightarrow x<z) \wedge (x<y \vee y<x \vee x=y))$
  2. "the order is dense"
     $(\forall x, y \, (x<y \Rightarrow \exists z \, (x<z<y)))$
  3. "the order has no max or min"
     (i.e. $\forall x \, \exists x_0, x_1 \, (x_0 < x \wedge x < x_1))$

E.g. if we interpret $<$ as the usual order on the rationals, then $(\mathbb{Q}, <) \models DLO$

likewise $(\mathbb{R}, <) \models DLO$

but $(\mathbb{Z}, <) \not\models DLO$ (not dense)

<u>Prop'n</u> DLO admits elim'n of quantifiers.

<u>Pf</u>: - we verify quant elim for simple formulas.

- atomic fmlas in this lang look like $y=z$ or $y<z$; negations like $y \neq z$ or $y \not< z$
- but observe $DLO \vdash y \neq z \Leftrightarrow y<z \vee y>z$

$DLO \vdash y \not< z \Leftrightarrow y>z \vee y=z$

- $\Rightarrow$ every simple fmla equiv to one of form $\exists x(\chi_1 \vee .. \vee \chi_k)$ where each χ_i is conjunction of atomic fmlas only (i.e. no negations)

(ex: if χ is e.g. $x \not< y \wedge x=z$ equiv to $(x>y \vee x=y) \wedge x=z$

$\underbrace{(x>y \wedge x=z)}_{\chi'} \vee \underbrace{(x=y \wedge x=z)}_{\chi''}$

so can replace w $\chi' \wedge \chi''$ if need be)

- Since $\exists x(\chi_1 \vee \cdots \vee \chi_k)$ equiv to $(\exists x\,\chi_1) \vee .. \vee (\exists x\,\chi_k)$, it's sufficient to show quant elim'n for formulas $\exists x\,\chi$ where χ is a conjunction of atomic fmlas, i.e. χ looks like:

$y_1<x \wedge .. \wedge y_n<x \wedge x<z_1 \wedge - \wedge x<z_m \wedge$
$x=w_1 \wedge .. \wedge x=w_p$

- Note: if any of the symbols y_i, z_j are x, fmla is false (i.e. equiv to $\perp$)

q.f. ☺

So assume this doesn't happen

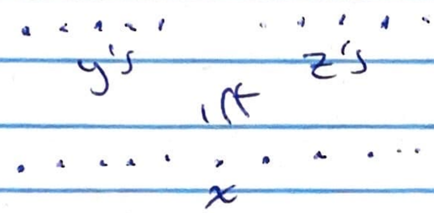
- likewise if any w_i is some y_i or z_j
- likewise if any y_i is some z_j
- i.e. we may assume ~~o~~ the symbol sets $\{y_i\}, \{z_j\}, \{w_k\}$ are disjoint and $x \notin \{y_i\} \cup \{z_j\}$

- now observe: if w is a variable not appearing in some fmla $\chi(x)$ ~~xxxx~~ ~~xxxxxxxxxxxxxxxxxxxx~~ then $\exists x\, \chi(x)$ is equiv to $\exists x (\chi(x) \wedge x = w)$

↳ ~~xxxxxxxxxx~~ above fmla is equiv to $y_1 < x \wedge .. \wedge y_n < x \wedge x < z_1 \wedge .. \wedge x < z_m$

- if there are no y's (i.e. $n=0$) then fmla asserts $x <$ all z_i
- such a fmla is true in all models of DLO, regardless of choice of z_i, since in such a model there's no left endpoint
- hence is provable from DLO (i.e. equiv to T, which is q.f.)
- likewise if there are no z's using that models have no right endpoint

- if there are both y's and z's then $\exists x \, (y_1 < x \wedge \dots \wedge y_n < x \wedge x < z_1 \wedge \dots \wedge x < z_m)$ is equivalent, because of density to the assertion that all y's are < all z's, i.e. $\chi' := (\bigwedge_{i,j} y_i < z_j)$ which is q.f. ✓



✓

- Observe that in passing from $\exists x \, \chi(x)$ to equiv q.f. formula χ' we eliminate variable x.
- if we begin w/ a <u>sentence</u> σ (i.e. no free vars), say in prenex form $Q_0 x_0 Q_1 x_1 \dots Q_n x_n \, \varphi(x_1, \dots, x_n)$ and eliminate innermost quantifier Q_n, we end up w/ equiv fmla $Q_0 x_0 \dots Q_{n-1} x_{n-1} \, \varphi'(x_0, \dots, x_{n-1})$ in which x does not appear.
- iterating this: the q.f. fmla we end up with after eliminating <u>all</u> quantifiers has no variables at all!

i.e. (if we assume in D.N.F.) will be of form $S_0 \vee S_1 \vee \dots \vee S_k$ where each S_i is either $\top$ or $\bot$.

- but then EZ to decide if entire disjunction is equiv to $\top$ (if one $\top$ appears) or $\bot$ (if none).
  ↳ this shows every sentence is provably equiv (from DLO) to either $\top$ or $\bot$!
- Since DLO consistent we have exactly one of DLO $\vdash \sigma$, DLO $\vdash \neg\sigma$ for every sentence σ.
  i.e.

Theorem DLO is complete, i.e. exactly one of σ and $\neg\sigma$ belongs to Conseq(DLO) for every sentence σ.

- It follows: Conseq(DLO) is decidable!
- Can be deduced from completeness using lemmas from 117b, or indeed directly: we've attained an algorithm above to decide if a given σ is (provably) equiv to $\bot$ or $\top$!

Ex: Consider the sentence

$$\exists y\, z\, \forall x\, (y<z \wedge (x \leq y \vee x \geq z))$$

- Asserts there is $y<z$ w/ no x between
  - would violate density
- Sentence is provably false from DLO but the point of above is: we can algorithmically show it is equiv to $\bot$ using elim of quantifiers

$$\Leftrightarrow \exists y, z\, \neg \exists x\, (y \geq z \vee \underbrace{(y<x \wedge x<z)}_{\text{by density equiv to just } y<z})$$

thus they equiv to $y \geq z \vee y<z$ which is $\top$

$\Leftrightarrow \exists y, z\, \neg \exists x\, \top$

$\Leftrightarrow \exists y, z\, \neg \top$

$\Leftrightarrow \exists y, y\, \bot \Leftrightarrow \bot$ ✓