- Might look like t.s. (if first compl is R, second L, last L):

$$r_{i_1}^{-1} l_{j_2}^{-1} \cdots l_{j_n}^{-1}\, t(\alpha,\beta)\, l_{j_n} \cdots l_{j_2} r_{i_1} = t$$

- Solving this eq'n for $t(\alpha,\beta)$ shows clearly it's in $\langle t\rangle' = \langle t, \{r_i\}, \{l_j\}\rangle$ ✓

Follows (from lemma + above): $T_M = \langle t\rangle' \cap G$

- Thus we've reduced halting prob of M, i.e. checking if a given (α,β) halts M, to checking if the element $t(\alpha,\beta) \in G'$ belongs to the subgroup $\langle t\rangle'$

- We can reduce this to word problem... for a slightly larger group.

<u>Theorem</u> There is a finitely presented group w/ unsolvable word problem.

<u>Pf</u>: Sps M has non-recursive halting set and let G' be our HNN ext

- Consider identity isomorphism of $\langle t \rangle'$. We take HNN ext of G_μ' by this iso!
- i.e. Consider G_μ'' defined by

$$G_\mu'' = \langle \{\text{generators for } G_\mu'\}, k \mid \{\text{relations for } G_\mu'\}, k^{-1}hk^{-1} = h \text{ for all } h \in \langle t \rangle' \rangle$$

- G_μ'' is finitely presented (since G_μ is and $\langle t \rangle$ finitely generated)
- By Britton's Lemma (see below), $\forall g \in G_\mu'$ we have $k^{-1}gk = g$ iff $g \in \langle t \rangle'$
- So: $k^{-1}t(\alpha,\beta)k = t(\alpha,\beta)$ iff $t(\alpha,\beta) \in \langle t \rangle'$ i.e. iff $t(\alpha,\beta) \in T_\mu$ iff μ halts on (α,β)

$\Rightarrow$ if we had algorithm deciding word prob for G_μ'', could decide halting set for μ — contradiction ✓

- Remains to prove Facts 1 + 2 there follow from "Britton's lemma"
- We sketch idea/proof of lemma, then deduce Facts 1+2 from it.

- Intuitively, Britton's lemma says HNN ext'ns are "as free as possible" wrt new conjugation relations
- What does this mean? We consider ext'ns by a single letter.

↳ Sps G a group, $\varphi: H \to K$ an isomorphism of subgroups, t new letter
- Let $G_\varphi := \langle G, t \mid t^{-1}ht = \varphi(h) \ \forall h \in H \rangle$ be HNN ext'n wrt φ.

- Can view G_φ as a quotient of the free product $G * \langle t \rangle$ by the relations $t^{-1}ht = \varphi(h)$
- words in $G * \langle t \rangle$ look like:

$$g_0 t^{e_1} g_1 t^{e_2} g_2 \cdots g_{n-1} t^{e_n} g_n$$

where $g_i \in G$, $e_i \in \{-1, 1\}$
- each el't of $G * \langle t \rangle$ can be uniquely written as such an "alternating product"

↳ which of these words in G_φ can be further reduced?

- two obv. ways such a word can be reduced:

① if tkt^{-1} appears in product (i.e. $e_i = 1$, $g_i = k$, $e_{i+1} = -1$) where $k \in K$, so that $k = t^{-1}ht$ for some $h \in H$

↳ can cancel tkt^{-1} to h then compute $g_{i-1} h g_{i+1}$ in G to get shorter word of same form

② if $t^{-1}ht$ appears where $h \in H$ so that $h = tkt^{-1}$ for some $k \in K$,

↳ can cancel $t^{-1}ht$ to k to get shorter word.

<u>Britton's Lemma</u> Given a word $g_0 t^{e_1} g_1 \cdots g_{n-1} t^{e_n} g_n$ (in $G * \langle t \rangle$), if we iteratively perform all reductions ①, ② above (say, starting from left), then whatever word $g_0' t^{e_1'} g_1' \cdots g_{k-1}' t^{e_k'} g_k'$ we end up with cannot be further reduced in G_e.

(somewhat imprecise statement - but approx. correct)

Pf: "exercise" (i.e. omitted, requires some insp.)

From Lemma, can quickly deduce Facts 1 + 2

<u>Fact 1</u>: if $g \in G$ then Lemma gives that the word g (viewed as in $G * \langle t \rangle$) cannot be further reduced in G_φ (i.e. to some other $g' \in G$ $g' \neq g$)

(might o.w. be conceivably possible, by adding some element/inverse pairs to the word g, involving the letter t, and reducing using $t^{-1}ht = \varphi(h)$ rel'ns)

$\Rightarrow$ The map $G \to G_\varphi$
$g \mapsto g$ is injective

$\Rightarrow$ G "is" a subgroup of G_φ. ✓

<u>Fact 2</u>: - Spse $L \leq G$ is a balanced subgroup of G: i.e. $\varphi(L \cap H) = L \cap K$

- Consider $L' = \langle L, t \rangle$ = subgrp in G_φ obtained by adding t to L.

- Claim is $L' \cap G = L$

$\supseteq$ is immediate

For $\subseteq$, concern would be if there is $g \in G \setminus L$ that can be computed (in G_φ) using only elements of L, and

- but this doesn't happen:
  ↳ sps $g_0 t^{e_1} g_1 t^{e_2} \cdots g_{n-1} t^{e_n} g_n$ is in G_φ w/ all $g_i \in L$ (all el'ts of L' can be written as such a product)

- Since $\varphi(L \cap H) = L \cap K$ (and hence also $\varphi^{-1}(L \cap K) = L \cap H$), if we can reduce any subwords tkt^{-1} to h then since $\varphi^{-1}(k) \in L$ $k \in L$ must have $h \in L$ too
- likewise if we can reduce any $t^{-1}ht$ to k, must have $k \in L$, since $h \in L$. $\varphi(h) \in L$

- Hence after reducing as much as possible we have word $g_0' t^{e_1'} \cdots g_{k-1}' t^{e_k'} g_k'$ we still have all $g_i' \in L$.

- only way such a word is in G is if $k = 0$, so word is g_0', which is in L by above. This gives $\subseteq$.

- Thus $G \cap L' = L$ ✓