- A word $v_0 v_1 \dots v_n$ is called <u>reduced</u> if there is no $i<n$ s.t. $v_i = v_{i+1}^{-1}$ or $v_i^{-1} = v_{i+1}$
- we wts if $v_0 v_1 \dots v_n$ is nonempty reduced word, then $\neq 1$ in G
- we sketch idea
- consider first two words $v_0 v_1$

$$= \underbrace{x^{-\alpha_0} y^{-\beta_0} t^{e_0} x^{\alpha_0} y^{\beta_0}}_{v_0} \underbrace{x^{-\alpha_1} y^{-\beta_1} t^{e_1} x^{\alpha_1} y^{\beta_1}}_{v_1}$$

- where $e_i \in \{1,-1\}$
- Since $v_0 \neq v_1^{-1}$ and vice versa must have either

① $(\alpha_0, \beta_0) \neq (\alpha_1, \beta_1)$ or

② $(\alpha_0, \beta_0) = (\alpha_1, \beta_1)$ and $e_0 = e_1$

- if ①, can reduce $v_0 v_1$ to

$$x^{-\alpha_0} y^{-\beta_0} t^{e_0} x^{(\alpha_0-\alpha_1)} y^{(\beta_0-\beta_1)} t^{e_1} x^{\alpha_1} y^{\beta_1}$$

at least one $\neq 0$

and no further reduction is possible by freeness between x, y and t

- if ② $v_0 v_1 = x^{-\alpha_0} y^{-\beta_0} t^{e_0} t^{e_0} x^{\alpha_0} y^{\beta_0}$

- inductively we see our reduced word $v_0 v_1 \dots v_n$ is $\neq 1$, since

after we perform cancellations between consecutive terms we still have #
of appearances of t^{e_i} for $e_i \in \{-1,1\}$
is $n+1$ ✓

- For every config $(\alpha,\beta) \in \mathbb{N}^2$ of our m.n. M
we now have the "code" $t(\alpha,\beta)$ in G
(and in fact in the subgroup T)
- we now code the instructions of M.

- idea: - for each instruct. $(a,b,c,R) \in M$
we consider isomorphism φ between
certain two subgroups of G, then pass
to HNN ext G_φ by letter r
  - symmetrically for (a,b,c,L) instructs
but now use ℓ as stable letter
  - Fact 2 $\Rightarrow$ can realize all these
extensions at once

- More explicitly: sps $(a,b,c,R) \in M$.
- says config $(uM+a, vM+b) \to_M (uM^2+cv)$

- in G want some corresponding way
of transforming
$t(uM+a, vM+b) =$
$x^{-uM}y^{-vM}(x^{-a}y^{-b}tx^ay^b)x^{uM}y^{vM}$

into $t(uM^2+c, v)$

$$= x^{-uM^2} y^{-v} (x^{-c} t x^{c}) x^{uM^2} y^{v}$$

- suggests we consider the assignment

$x^M \mapsto x^{M^2}$

$y^M \mapsto y$

$t(a,b) \mapsto t(c,0)$

- and indeed not hard to convince self this extends to an isomorphism

$\varphi : G^{M,M}_{c,b} \to G^{M^2,1}_{c,0}$

between the subgroups

$G^{M,M}_{a,b} = \langle x^M, y^M, t(a,b)\rangle$

$G^{M^2,1}_{c,0} = \langle x^{M^2}, y, t(c,0)\rangle$

since both of these subgroups iso to $\mathbb{Z} * \mathbb{Z}^2$ as well.

- likewise if $(a, b, c, L) \in \mathcal{M}$ says that ~~[illegible]~~ $(uM+a, vM+b) \to (u, vM^2+c)$ we consider the natural isomorphism

$\chi : G^{M,M}_{a,b} \to G^{1,M^2}_{0,c} = \langle x, y^{M^2}, t(0,c)\rangle$

given by extending

$x^M \mapsto x \quad y^M \to y^{M^2} \quad t(a,b) \to t(0,c)$

(how to justify e.g. $x \to x^M$ $y \to y^M$ $t \to t(a,b)$ extends to wo from G onto $G^{MM}_{a,b}$?

- in G every el't g can be put uniquely in form $x^{k_0} y^{l_0} t^{m_0} x^{k_1} y^{l_1} t^{m_1} \dots x^{k_n} y^{l_n} t^{m_n}$
- in $G^{MM}_{a,b}$ g maps to $x^{Mk_0} y^{Ml_0} t(a,b)^{m_0} \dots x^{Mk_n} y^{Ml_n} t(a,b)^{m_n}$
- can further reduce this word in G (terms $t(a,b)^{m_i} = x^{-a_i} y^{-b_i} t^{m_i} x^{a_i} y^{b_i}$ will combine w/ x^{Mk_i}'s and y^{Ml_i}'s etc.) but can check once reduced we can recover original exponents of x^M's y^M's and $t(a,b)$'s if we knew M, a,b)

Thus $x^M \to x^{M^2}$ $y^M \to y$ $t(a,b) \to t(c,0)$ likewise defines canonical wo $G^{MM}_{a,b} \to G^{M^2,1}_{c,0}$

Pov: - thinking of each $t(\alpha,\beta)$ as a code for the config (α,β), then if $(\alpha,\beta) = (uM+a, vM+b)$ is a config to which an instruction (a,b,c,D) applies then $G^{M,M}_{a,b}$ contains all the codes $t(\alpha,\beta)$ for such configs

and $G^{M^2,1}_{c,0}$, $G^{1,M^2}_{0,c}$ contain all codes $t(\alpha',\beta')$ for configs

$(\alpha', \beta') \Leftarrow (uM^2 + c, v)$
or $(u, vM^2 + c)$ depending (if $D = R$ or L)

obtained by applying such instructions

- Note: these subgroups fin. gen.

↳ makes sense to consider the related subgroups of T $(= \langle t(\alpha,\beta) \rangle)$ generated only by such codes

Def'n. For $a, b < M$ let $T^{M,M}_{a,b} = \langle t(uM+a, vM+b) : u, v \in \mathbb{Z} \rangle$
$= \langle x^{-uM} y^{-vM} t(a,b) x^{uM} y^{vM} : u, v \in \mathbb{Z} \rangle$
= grps generated by the $t(\alpha,\beta)$'s which code configs $(\alpha, \beta) = (uM+a, vM+b)$ that instructions of form (a, b, c, D) apply to

↳ the $T^{M,M}_{a,b}$'s not f. in gen (they're freely generated subgroups of $T = \langle t(\alpha,\beta) : \alpha, \beta \in \mathbb{Z} \rangle$)

Lemma $T^{M,M}_{a,b} = T \cap G^{M,M}_{a,b}$
(i.e. el'ts of $G^{M,M}_{a,b}$ in T are exactly those in $T^{M,M}_{a,b}$)

Pf: not hard, leave as exercise.
moral is: no codes in $G^{M,M}_{a,b}$ that shouldn't be there $\hat{t}(a,b)$ ✓

$\Rightarrow$ lemma gives that T is balanced wrt canonical isos between $G^{M,M}_{a,b} \to G^{M^2,1}_{c,c}$
and " $\to G^{1,M^2}_{c,c}$

Why: call iso φ.
then $\varphi(T \cap G^{M,M}_{a,b}) \overset{\text{lemma}}{=} \varphi(T^{M,M}_{a,b})$
$= T^{M,M^2}_{c,c}$ (since $\varphi(t(a,b)) = t(c,c)$)
$= T \cap G^{M^2,1}_{c,c}$
$= T \cap \varphi(G^{M,M}_{a,b})$ ✓

likewise for left ones

- Now: given m.m. $\mathcal{M}$ with $M > 1$ and instructions $\{(a_i, b_i, c_i, R) : i \in I\}$ $\cup \{(a_j, b_j, c_j, L) : j \in J\}$ we define an extension $G_{\mathcal{M}}'$ of G ($'$ indicates HNN extension)

- for each $i \in I$, let φ_i denote canonical iso from $G^{M,M}_{a_i,b_i} \to G^{M^2,1}_{c_i,c_i}$

$= \varphi(x^M)$

- introduce letter r_i, add to generators of G, along w/ rules $r_i^{-1} x^M r_i = x^{M^2}$

$$r_i^{-1} y^M r_i = y^M$$
$$r_i^{-1} t(a,b) r_i = t(c,c)$$

- since $x^M, y^M, t(a,b)$ generate $G^{M,M}_{a,b}$, these imply $r_i^{-1} g r_i = \varphi(g)$ $\forall g \in$ " "

- likewise for $j \in J$ let χ_j denote iso $G^{M,M}_{e_j,b} \to G^{1,M^2}_{c,c}$, add letter l_j and conjugation rules for l_j corr'ing to χ_j

- let G_n' be HNN extension of G defined by presentation $\langle t, x, y, \{r_i\}, \{l_j\} \mid xy = yx$ [.. conjug. rules for r_i, l_j ..] $\rangle$

- G is subgroup of G_n' (Fact 1) and ~~[illegible]~~ conjugation by each r_i, l_j on G_n' extends isomorphisms φ_i, χ_j

- As observed, prev. lemma $\Rightarrow T$ is balanced subgroup of G_n'

$\Rightarrow$ (Fact 2) $T = T' \cap G$

(where T' = HNN ext. of T viewed as subext of G_n', i.e. subgroup obtained by adding new letters r_i, l_j to T)

- Consider now the following subgroup of T:

$$T_\mu := \langle t(\alpha,\beta) : (\alpha,\beta) \in H_\mu \rangle$$

where H_μ = halting set of μ.
(i.e. T is (freely) generated by codes $t(\alpha,\beta)$ for halting inputs (α,β) of M)

- T_μ is also balanced in G'_μ
- Why: key point is: if e.g. $e_i(t(\alpha,\beta)) = (\alpha_1,\beta_1)$ then $(\alpha,\beta) \to_\mu (\alpha_1,\beta_1)$ in μ (by instruct corr'ing to e_i); likewise for left instruct
- When $(\alpha,\beta) \to_\mu (\alpha_1,\beta_1)$ we have $(\alpha\beta) \in H_\mu \Leftrightarrow (\alpha_1,\beta_1) \in H_\mu$ (how computation of μ works), and hence $t(\alpha,\beta) \in T_\mu$ iff $t(\alpha_1,\beta_1) \in T_\mu$.
- together this gives: each e_i maps all halting codes in $G^{M,M}_{e_i,b}$ onto halting codes in $G^{M,M}_{\hat{e}_i,c}$ (likewise for non-halting codes; likewise for the x_j) — this is balancedness.

$\Rightarrow$ (Fact 2) $T_\mu = T'_\mu \cap G$ (i.e. if we add r_i, e_j's to T_μ we don't generate (in the extension) any codes $t(\alpha,\beta)$ for configs (α,β) that don't halt in M.

Crucial Lemma: $T_M' = \langle t\rangle'$, (i.e. if we add new letters r_i, l_j to free subgroup $\langle t\rangle$ in resulting extension $\langle t\rangle'$ get all halting codes $t(\alpha,\beta)$ — and no non-halting codes)

Pf: ($\supseteq$) immediate: $t = t(0,0)$ is code for $(0,0) \in H_M$ so $t \in T_M \Rightarrow \langle t\rangle' \subseteq T_M'$

($\subseteq$) also clear once we reflect; I'll just describe idea.

- point is: the wo's u_i, x_j code computations of M; in G_M' the stable letters r_i, l_j effect these wo's by global conjugations
- given a sequence of computations

$$(\alpha,\beta) \to_M (\alpha_1,\beta_1) \to (\alpha_2,\beta_2) \to \dots \to (\alpha_n,\beta_n) = (0,0)$$

witnessing $(\alpha,\beta) \in H_M$

there is a corr'ing seq. of conjugations of $t(\alpha,\beta)$ yielding $t = t(0,0)$