<u>Theorem</u> There exists an unsolvable semi-Thue process Π (in some alphabet A)

<u>Pf</u>: - For a Turing machine M, we find a process Π_M s.t. halting prob'm for M reduces to word prob for Π_M.

- Consider machines M in alphabet $\{1\}$
- 0 represents empty square of tape.
- Let $Q_0, \ldots, Q_N$ denote states of M w/ Q_0 initial state, Q_N terminal
- for each Q_i, $i<N$, and $a \in \{0,1\}$ we have an instruction (Q_i, a, b, D, Q_j) where: $b \in \{0,1\}$, $D \in \{\text{left}, \text{right}\}$, and $j \leq N$
  ↳ "if in state Q_i and square reads a, print b, move D, go to state Q_j"
- if we ever reach Q_N, we halt.
- if input word is $1^{(n)} = \underbrace{11\cdots1}_{n\text{-times}}$

then tape reads $\ldots 001^{(n)}00 \ldots$

- machine starts at leftmost 1 in state Q_0 and follows instructions until halt (or indefinitely)

(- if $n=0$, tape is all 0s, begin at any 0 in state Q_0)

- We define a semi-Thue process on alphabet $\{0, 1, Q_0, \ldots, Q_N, Q, Q', \#\}$
- Consider words of form $\# u Q_i v \#$ w/ $u, v \in \{0,1\}^*$
- represents situation in which M has w printed in already explored part of tape (0's beyond the #'s), is in state Q_i, read/write head on first symbol of v

- Process Π_M consists of productions of four types:

① for all instructions $(Q_i, a, b, \text{right}, Q_j)$ includes:

$$Q_i a 0 \to b Q_j 0$$
$$Q_i a 1 \to b Q_j 1$$
$$Q_i a \# \to b Q_j 0 \#$$

② " " $(Q_i, a, b, \text{left}, Q_j)$ includes:

$$0 Q_i a \to Q_j 0 b$$
$$1 Q_i a \to Q_j 1 b$$
$$\# Q_j a \to \# Q_j 0 b$$

③ includes: $Q_N 0 \to Q 0$ and $Q_N 1 \to Q 1$

④ includes:

$$Q 0 \to Q$$
$$Q 1 \to Q$$
$$Q \# \to Q' \#$$
$$0 Q' \to Q'$$
$$1 Q' \to Q'$$

<u>Observe</u>: if M halts @ input $1^{(n)}$ with uv on explored part of tape and r/w head at v, follows from above:

$$\begin{aligned} \# Q_0 1^{(n)} \# &\Rightarrow^*_{\Pi_M} \# u Q_N v \# \\ &\Rightarrow^*_{\Pi_M} \# u Q v \# \\ &\Rightarrow^*_{\Pi_M} \# u Q \# \\ &\Rightarrow^*_{\Pi_M} \# Q' \# . \end{aligned}$$

Conversely, we claim that if $\# Q_0 1^{(n)} \# \Rightarrow^*_{\Pi_M} \# Q' \#$ then M halts on $1^{(n)}$

<u>Pf</u>. - the only way to reach $\# Q' \#$ is by applying productions 3 and 4 to a word of form $\# u Q_N v \#$
- Backtracking to this step, we see we have $\# Q_0 1^{(n)} \# \Rightarrow \# u Q_N v \#$
- Sequence of productions witnessing this can only be from 1 and 2 (once Q, Q' introduced, can't go back)
- but these correspond to computations of M
- hence M reaches Q_N on input $1^{(n)}$, i.e. terminates ✓

- Now: let M be a Turing machine whose halting problem is undecidable (e.g. universal t.m.)
- if Π_M ~~[illegible]~~ solvable, could use algorithm to compute halting set of M
  $\Rightarrow$ Π_M not solvable ✓

Thue Processes

Def'n a Thue process Π (on alphabet A) is a semi-Thue process closed under inverses, i.e. if $w \to w' \in \Pi$
then inverse $w' \to w \in \Pi$ too.

- If Π is Thue proc. then $u \Rightarrow_\Pi^* v$ implies $v \Rightarrow_\Pi^* u$
- we get equiv. relation on words in A^*: write $u \sim v$ iff $u \Rightarrow v$ (iff $v \Rightarrow u$)
- write $[u]$ for equiv class:
  $[u] = \{v \in A^* \mid u \sim v\}$
- Can form a semigroup as follows:
- Let $S = S_\Pi = A^*/\sim = \{[u] : u \in A^*\}$
- Notice: since
  $u \sim u'$ and $v \sim v'$ implies $uv \sim u'v'$

We can define an (associative) product $*$ on S by:

$$[u] * [v] = [uv]$$

- then $(S, *)$ is a semigroup w/ identity $[\emptyset]$
- Conversely, note that every semigroup presentation corresponds to a Thue process Π, and S_Π is the semigroup determined by presentation
- (So Thue processes formalize concept of "semigroup presentation")

<u>Theorem</u> (Post, Markov) There is a semigroup S (w/ finite presentation) whose word problem is undecidable.

<u>Pf.</u> - as in prev. thm, given tm M, let Π_M be assoc. semi-Thue process
- Let Π_M' be Thue process obtained by adding inverses to Π_M
- We showed:

$$M \text{ halts on } 1^{(n)} \text{ iff } \# Q_0 1^{(n)} \# \Rightarrow_{\Pi_M} \# Q' \#$$

- If $\# Q_0 1^{(n)} \# \Rightarrow_{\Pi_M} \# Q' \#$ then cfc

$$\# Q_0 1^{(n)} \# \Rightarrow_{\Pi_M'} \# Q' \#$$

We claim converse also holds

Pf: - Sps $\#Q_0 1^{(n)} \# \Rightarrow_{\Pi_M'} \# Q' \#$

- Let $w_0 \Rightarrow_{P_0'} w_1 \Rightarrow_{P_1'} \cdots \Rightarrow_{P_{n-1}'} w_n$ be a sequence of substitutions witnessing this, with $P_i' \in \Pi_M'$ $\forall i < n$

So:
$w_0 = \#Q_0 1^{(n)} \#$
$w_n = \#Q' \#$

- We find a sequence in Π_M turning w_0 into w_n

- Let i be maximal s.t. $P_i' \notin \Pi_M$
- must have $i < n-1$: there is no production in Π_M transforming $\#Q'\#$ hence none in Π_M' transforming into $\#Q'\#$

- hence we have
$$w_i \Rightarrow_{P_i'} w_{i+1} \Rightarrow_{P_{i+1}'} w_{i+2}$$
w/ P_i' = inverse of some $P \in \Pi_M$
P_{i+1}' = some $R \in \Pi_M$

- we write:
$$w_i \;{}_P\!\Leftarrow w_{i+1} \Rightarrow_R w_{i+2}$$

- Observe: w_{i+1} is a word of the form $\#uQ_jv\#$ or $\#uQv\#$ or $\#uQ'\#$

- by inspection of production in Π_M exactly one applies to w_{i+1}

- thus $P = R$ which gives $w_i = w_{i+2}$
- by induction can remove all P_i not in Π_M ✓

- Follows: a universal t.m. M halts on $1^{(n)}$ iff $\# Q_0 1^{(n)} \# \Rightarrow_{\Pi_M'} \# Q' \#$
iff $[\# Q_0 1^{(n)}] = [\# Q' \#]$
in the semigroup $S = S_{\Pi_M'}$.

→ S's word problem is unsolvable (since halting problem for M undec). ✓

Word problem for groups

- Semigroup S constructed above w/ unsolvable word problem is not a group. (productions include no product inverse rules)

- We can associate a Thue process to a group presentation just like to a semigroup presentation

<u>Setup</u>: Given $A = \{a_1, \dots, a_n\}$ and relations $R_1, \dots, R_k$ where R_i of form $w_i = v_i$, the group presentation

$$\langle a_1, \dots, a_n \mid R_1, \dots, R_k \rangle$$

determines a Thue process Π on $A \cup A^{-1} = \{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}$ w/ productions $w_i \to v_i$ for every R_i

+ inverse productions of these
+ productions $a_i a_i^{-1} \to \emptyset$
$a_i^{-1} a_i \to \emptyset$
+ inverses of these.

- then associated semigroup $G = (A \cup A^{-1})^* / \sim$ is a group: "the group determined by the presentation"

- We want to prove:

<u>Theorem</u> (Boone, Novikov) There is a (finitely presented) group whose word problem is undecidable

i.e. an alphabet A and relations R_i whose associated Thue process (in group sense) has unsolvable word problem.

↳ to prove, will need to develop some group theory + introduce modular machines.

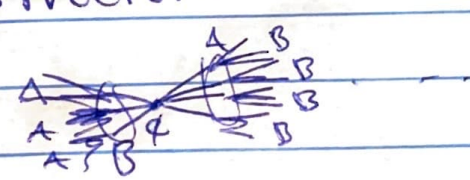
Groups and free products

- Sps Γ_A and Γ_B are groups given by:

$$\Gamma_A = \langle \overbrace{a_1, \dots, a_n}^{A} \mid R_1, \dots, R_k \rangle$$
$$\Gamma_B = \langle \underbrace{b_1, \dots, b_m}_{B} \mid S_1, \dots, S_\ell \rangle$$

- We assume $A \cap B = \emptyset$.
- the <u>free product</u> of Γ_A, Γ_B is given by:

$$\Gamma_A * \Gamma_B = \langle a_1, \dots, a_n, b_1, \dots, b_m \mid R_1, \dots R_k, S_1, \dots, S_\ell \rangle$$

- Observe: every el't in $\Gamma_A * \Gamma_B$ is (equiv to) a word of form $w_0 w_1 w_2 \cdots w_n$ w/ each w_i in either Γ_A or Γ_B and we alternate groups from i to $i+1$

- tree like structure: