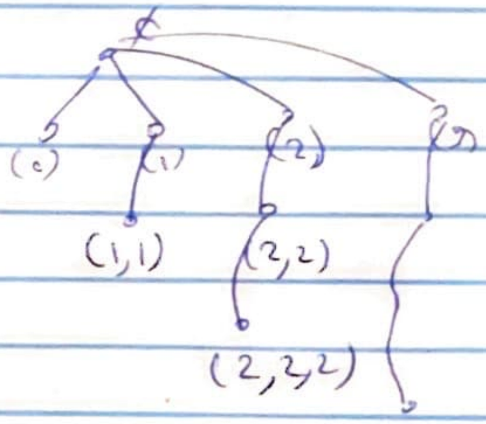
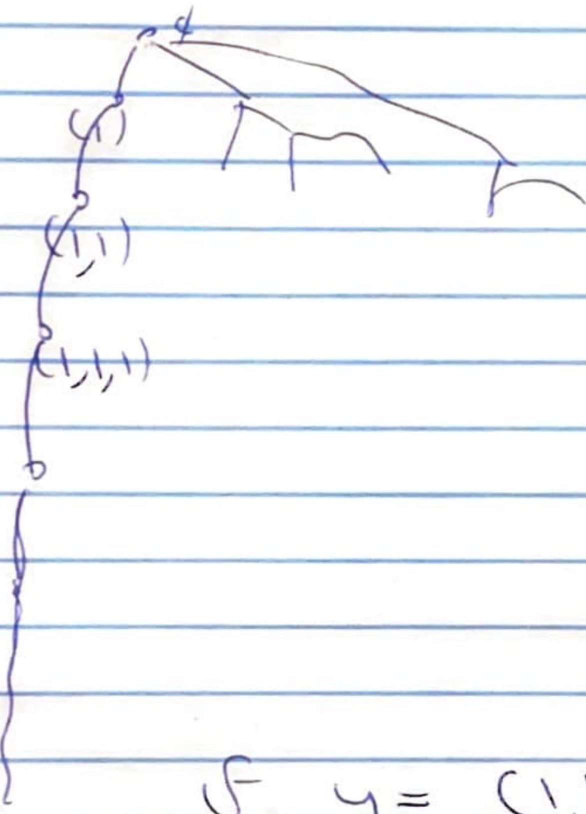
T



longer + longer finite branches → but no infinite ones

but some trees have infinite branches.

S



if $y = (1,1,1,\dots) \in \mathbb{N}^{\mathbb{N}}$
then $y \restriction n \in S$ for every $n \in \mathbb{N}$

Def'n A (non-strict) <u>linear order</u> (on $\mathbb{N}$) is a binary relation $\leq$ (i.e. a subset of $\mathbb{N}\times\mathbb{N}$) that is antisymmetric, transitive, and total.

- We can represent a linear order visually as the points in $\mathbb{N}$ arranged in a line in some way:

e.g. $0 \leq 1 \leq 2 \leq \cdots$ is a l.o. on $\mathbb{N}$ (the usual one)

$\circ \quad \circ \quad \circ \quad \cdots$

so is $0 \leq 2 \leq 4 \leq \cdots \leq 1 \leq 3 \leq 5 \leq \cdots$

$\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \quad \circ \quad \cdots$

so is: $\cdots \leq 4 \leq 2 \leq 0 \leq 1 \leq 3 \leq 5 \leq \cdots$

$\cdots \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \cdots$

A linear order is called a <u>well-order</u> if it does not contain an infinite strictly decreasing sequence

e.g. first two ex's above are well-orders but third is not since $0 \geq 2 \geq 4 \geq \cdots$ is a strictly decreasing sequence in this order.

(i)

- Given $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we say $x \in \mathbb{N}^{\mathbb{N}^{<\mathbb{N}}}$ (viewed as a copy of $\mathbb{N}^{\mathbb{N}} = \mathcal{N}$) is a <u>code for T</u> iff, $\forall s \in \mathbb{N}^{<\mathbb{N}}$, $x(s) = 1$ iff $s \in T$
- Say $x = x_T$ is <u>the</u> code for T if furthermore $x(s) = 0$ when $s \notin T$
- Conversely given $x \in \mathbb{N}^{\mathbb{N}^{<\mathbb{N}}}$, define $T_x :=$ subset of $\mathbb{N}^{<\mathbb{N}}$ coded by $x := \{s \in \mathbb{N}^{<\mathbb{N}} : x(s) = 1\}$

- Define $WF \subseteq \mathcal{N}$ $(= \mathbb{N}^{\mathbb{N}^{<\mathbb{N}}})$:= set of codes for <u>well-founded</u> trees

- Likewise, given $L \subseteq \mathbb{N} \times \mathbb{N}$, say $x \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}}$ (also viewed as a copy of $\mathcal{N}$) is a <u>code for L</u> if $x(n,m) = 1$ iff $(n,m) \in L$

- Define $WO \subseteq \mathcal{N}$ $(= \mathbb{N}^{\mathbb{N} \times \mathbb{N}})$:= set of codes for ctbl <u>well-orders</u>

- Now viewing WF, WO as subsets of $\mathcal{N}$, can ask: how complicated are these sets, i.e. where do they fall in the arithmetical / analytical hierarchy?

(ii)

- Intuitively: both are above the hyperarithmetical, since they require second order properties to define:
  - ↳ "no infinite branch"
  - "no infinite descending sequence"
  - ↳ these refer to subsets of the universe of the structure ⇒ need $\exists^{2}$ or $\forall^{2}$ quantifiers to define.

<u>Prop'n</u> WF and WO are Π^1_1

Pf: - Note $x \in WF$ iff conjunction of following fmlas hold:

① $\forall s, t \in \mathbb{N}^{<\mathbb{N}}$ ("$= {}^{<}\mathbb{N}$")
$(s \in T_x \wedge t < s \Rightarrow t \in T_x)$
(says "T_x is a tree")

here $t<s$ = t initial segment of s.

② $\neg(\exists y \in \mathcal{N}) \forall n \in \mathbb{N} (y \restriction n \in T_x)$
("T_x is well-founded")

⇒ It follows: WF is Π^1_1!
But why?

↳ we can analyze ① and ② to understand subset of $\mathcal{N}$ their conjunction defines.

(iii)

Idea: • look @ q.f. parts + determine complexity of defined set (often effective open (Σ^0_1) or even clopen (Δ^0_1))
$\wedge$'s are $\cap$'s and $\vee$'s are $\cup$'s
• then @ quantifiers: $\exists^{\mathbb{N}}, \forall^{\mathbb{N}}$ are (effective) union/intersection over index $\mathbb{N}$ (or some power); $\exists^{\mathcal{N}}, \forall^{\mathcal{N}}$ effective $\cup/\cap$ over index $\mathcal{N}$
$\Rightarrow$ each alternating $\exists^{\mathbb{N}}, \forall^{\mathbb{N}}$ brings up hyperarithmetical complexity
$\exists^{\mathcal{N}}, \forall^{\mathcal{N}}$ bring us into analytical hierarchy.

– e.g. consider q.f. part of ①
– rewritten it's: $(x(s) = 1 \wedge t < s) \Rightarrow x(t) = 1$
(– imagine $s, t \in \mathbb{N}^{<\mathbb{N}}$ as fixed for now)
– equivalently, using just $\wedge$'s and $\vee$'s it's: $x(s) \neq 1 \vee \neg(t<s) \vee x(t) = 1$

Q's: what's complexity of subset of $\mathcal{N}$ (viewed as $\mathbb{N}^{\mathbb{N}^{<\mathbb{N}}}$) defined by ①? i.e. the set $\{x \in \mathcal{N} : x(s) \neq 1 \vee \neg(t<s) \vee x(t) = 1\}$

– is same as $\{x : x(s) \neq 1\} \cup \{\neg(t<s)\} \cup \{x : x(t) = 1\}$

(iv.)

- third set is (basic) open:
  ↳ tho we defined basic open subsets (of $\mathbb{N}^{\mathbb{N}} = \mathcal{N}$) as the $N_s := \{x : s < x\}$, it's equiv (and easier when working w/ $\mathcal{N}$ as $\mathbb{N}^{\mathbb{N}^{<\mathbb{N}}}$) to define the basic open sets (given $s \in \mathbb{N}^{<\mathbb{N}}$, $k \in \mathbb{N}$) as $N_{s,k} = \{x : x(s) = k\}$
- third set is $N_{t,1}$ (hence complexity is Σ^0_1 - in fact Δ^0_1)

- first set is the union
  $\{x : x(s) = 0\} \cup \{x : x(s) = 2\} \cup \{x : x(s) = 3\} \cup \ldots$

- $\bigcup_{n \neq 1} N_{s,n} \Rightarrow \Sigma^0_1$

- middle set is either $\mathcal{N}$ or $\emptyset$ (depending on whether $\neg(t < s)$ or $t < s$)

- thus first $\cup$ second $\cup$ third is
  $\Sigma^0_1 \cup \Sigma^0_1 \cup \Sigma^0_1 = \Sigma^0_1$
- the leading $\forall s, t$ then takes an (effective) intersection over all pairs $s, t \in \mathbb{N}^{<\mathbb{N}}$ of a Σ^0_1 set giving a Π^0_1 set.
  $\Rightarrow$ subset defined by ① is Π^0_1
  $\subseteq \mathcal{N}$

(v.)

- Likewise w/ (2): can analyze q.f. part along w/ leading $\exists n \in \mathbb{N}$ quantifier will define an arithmetical set (prob Σ^0_1)
- but now: leading $\exists y \in \mathcal{N}$ quantifier indicates union indexed by $\mathcal{N}$: defines a Σ^1_1 set
- then $\neg$ changes complexity to Π^1_1

$\Rightarrow$ (1) $\wedge$ (2) defines a Π^1_1 set
i.e. $\mathrm{WF} \subseteq \mathcal{N}$ is Π^1_1 ✓

Similarly, for $\mathrm{WO} \overset{\subseteq \mathcal{N} = \mathbb{N}^{\mathbb{N}\times\mathbb{N}}}{}$ we have $x \in \mathrm{WO}$ iff following conjunction holds:

(1.) $\forall n, m \in \mathbb{N} \; [(n \leq_x m \wedge m \leq_x n) \Rightarrow n = m]$
"$\leq_x$ is antisymmetric"

(2.) $\forall n, m, p \in \mathbb{N} \; [(n \leq_x m \wedge m \leq_x p) \Rightarrow n \leq_x p]$
("$\leq_x$ is transitive")

(3.) $\forall n, m \in \mathbb{N} \; [n \leq_x m \vee m \leq_x n]$
("$\leq_x$ is total")

(4.) $\forall y \in \mathcal{N} \; \exists n \in \mathbb{N} \; [y(n) \leq_x y(n+1)]$
("$\leq_x$ is a well-order")

$\Rightarrow$ It follows by a similar analysis: WO is Π^1_1

(VI.)

- Natural Q: can we rule out there isn't some simpler def'n of either WF or WO (i.e. both belong to $\Pi^1_1 \setminus \Delta^1_1$)?
- We can!

__Def'n__ Let $X = \mathbb{N}^k \times \mathcal{N}^\ell$. A set $A \subset X$ is __Π^1_1-complete__ if it is Π^1_1 and for any Π^1_1 set $B \subseteq Y$ (for any Y of form $\mathbb{N}^k \times \mathcal{N}^\ell$) there is a computable $f : Y \to X$ s.t.

$$x \in B \iff f(x) \in A$$

(as usual: means checking membership in A at least as hard as checking in B)
(such f called a __reduction__ of B to A)

__Theorem__ WF and WO are Π^1_1-complete.

- Before we prove, need to discuss __rep'ing__ Π^1_1 __sets__ as __bodies__ __of__ __computable trees__.

- To every subset $B \subseteq \mathcal{N}$ can assoc. a tree $T_B \subseteq \mathbb{N}^{<\mathbb{N}}$, defined by

(VII)

$T_B = \{s \in \mathbb{N}^{<\mathbb{N}} : \exists x \in B,\ s \prec x\}$

- then every $x \in B$ is a branch of T_B
- if we write $[T]$ for set of branches of a tree T, this says $B \subseteq [T_B]$
- $\supseteq$ not always true
  e.g. if $B = \{0111..,\ 00111..,\ 000111..\}$
  then $000.. \in [T_B]$ but not in B.
- but $\supseteq$ is true (exercise!) when B is closed (i.e. Π^0_1)

- Conversely can check: if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree then $[T]$ is closed.
- This characterizes closed (i.e. Π^0_1) sets: B is Π^0_1 iff $\exists$ a tree T s.t. $B = [T]$.

- not too hard to check effective version of this is also true: i.e. B is Π^0_1 (effect. closed) iff there is a computable tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ s.t. $B = [T]$.
- same for $B \subseteq \mathcal{N}^k$

- from this, get a rep'n of Σ^1_1 sets (and Π^1_1 sets, by complements)

(viii)

- If $A \in \Sigma^1_1$ then is proj. along $\mathcal{N}$ of Π^0_1 set, i.e. there is $S \in \Pi^0_1$ s.t. $x \in A$ $\Leftrightarrow \exists y \in \mathcal{N}((x,y) \in S)$
- But $S = [T]$ for some computable T!
- thus we get

  $x \in A \Leftrightarrow \exists y \in \mathcal{N}((x,y) \in [T])$

$\hookrightarrow$ will use this to prove Π^1_1 completeness of WO/WF

Pf: - Already shown WF WO $\in \Pi^1_1$
- show WF is Π^1_1 - complete first.
- Fix $B \subseteq \mathcal{N}$ that is Π^1_1
- we reduce B to WF (can check: suffices to find reductions for Π^1_1 subsets of $\mathcal{N}$)

- Fix a computable T on $\mathbb{N} \times \mathbb{N}$ s.t.
$x \notin B \Leftrightarrow \exists y \in \mathcal{N}((x,y) \in [T])$
(B is complement of a Σ^1_1 set!)

- $T \subseteq (\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$, i.e. consists of finite sequences of pairs, i.e. is a "tree of pairs"
- but we usually think of T as a "pair of trees"

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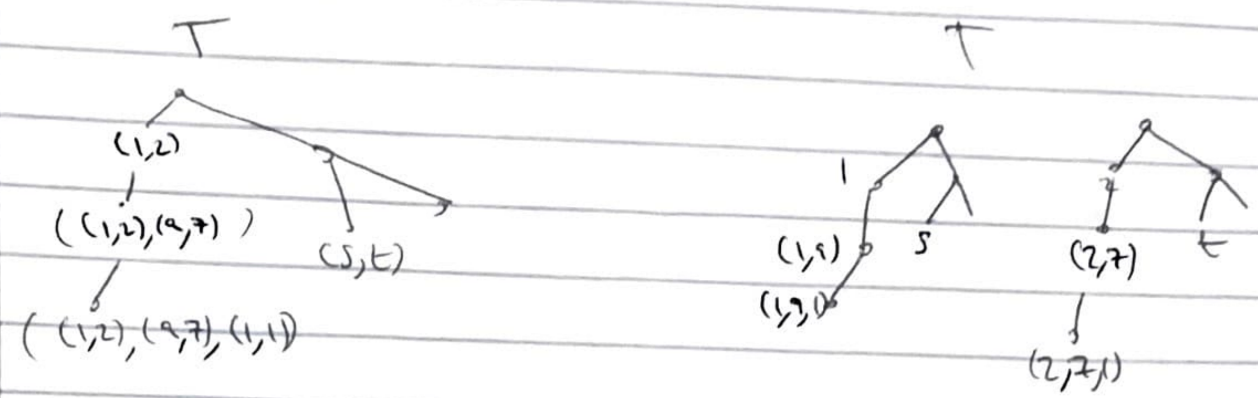
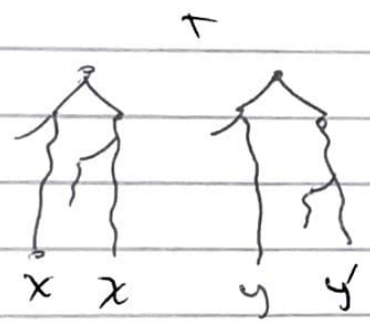


- think of el'ts of T as a pair of sequences (s,t) (of same length)

- left and right parts of T have same shape
- needn't be bonafide trees on $\mathcal{N}$: diff is, given $x \in \mathcal{N}^{\mathcal{N}}$ could appear as branch multiple times in (say) lefthand tree



but then corr. branches on right are distinct sequences

- w/ this in mind define: given $x \in \mathcal{N}$
$T(x) := \{(s,t) : s \prec x\}$

- ez to check: $T(x)$ is subtree of T and is w.f. iff x does not appear

$\otimes$

as branch in lefthand part of T
iff $\nexists y \in \mathcal{N}\,((x,y) \in [T])$
iff $x \in B$

- but then: map that takes $x \to T(x)$
(which is computable, since T is)
reduces B to WF
  (technically, should be working w/
  codes for the $T(x)$'s)

$\Rightarrow$ WF is Π^1_1-complete. ✓

- To prove WO is Π^1_1-complete
we reduce WF to it.
- make use of Kleene-Brouwer
ordering: given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$
define an ordering $<^T_{KB}$ on T
by: $s <^T_{KB} t$ iff either
  - $s > t$, or
  - neither s,t extends the
other and $s <_{lex} t$, i.e. $s(n) < t(n)$
where n is least s.t. $s(n) \neq t(n)$

- Can check: $<^T_{KB}$ is a linear order
and moreover is a well-order
iff T is well-founded