Pf. ($\Leftarrow$) is immediate
($\Rightarrow$) - we'll do case when $X = \mathcal{N}$
- if A is Σ^0_1 then $A = \bigcup_{s \in B} N_s$ for some $B \subseteq \mathbb{N}^{<\mathbb{N}}$
- But then A is $\Sigma^0_1(B)$ (i.e. $\Sigma^0_1(x)$ where $x \in \mathcal{N}$ is the el-sequence that codes B) ✓

The arithmetical hierarchy

- Can now officially define the arithmetical hierarchy (for subsets of $\mathbb{N}^k \times \mathcal{N}^\ell$) which generalizes def'n we gave in 117b (for $\mathbb{N}^k$)

Def'n The pointclasses $\Sigma^0_n, \Pi^0_n, \Delta^0_n$ for all spaces of the form $\mathbb{N}^k \times \mathcal{N}^\ell$ are defined inductively by:

(i) A set $A \subseteq \mathbb{N}^k \times \mathcal{N}^\ell$ (for some k, ℓ) is Σ^0_1 iff it is effectively open
(2) $A \in \Pi^0_1$ iff complement of A is Σ^0_1
(3) $A \in \Sigma^0_{n+1}$ iff it is the "projection along $\mathbb{N}$" of a Π^0_n set, i.e.

iff there is a Σ^0_n set $B \subseteq \mathbb{N}^{k+1} \times \mathcal{N}^\ell$
s.t. $x \in A \Leftrightarrow \exists m \in \mathbb{N}\ ((m,x) \in B)$

(4) $A \in \Delta^0_n$ iff A is both Σ^0_n and Π^0_n

A set in one of these pointclasses is called an <u>arithmetical set</u>.

A note on part (3.) of def'n:

- it's an exercise to check that if $B \subseteq \mathbb{N}^{k+1} \times \mathcal{N}^\ell$ is Π^0_n then for any fixed $m \in \mathbb{N}$, the fiber $\{x \in \mathbb{N}^k \times \mathcal{N}^\ell : (m,x) \in B\}$ is also Π^0_n. Write B_m for this fiber.
- the condition $\exists m \in \mathbb{N}((m,x) \in B)$ is same as $\exists m \in \mathbb{N}\,(x \in B_m)$
- i.e. in (3.) we have $A = \bigcup_{m \in \mathbb{N}} B_m$ is an "effective" union of Π^0_n sets

Since Σ^0_1 sets over spaces $\mathbb{N}^k \times \mathcal{N}^\ell$ generalize these over spaces $\mathbb{N}^k$ (i.e. the Σ^0_1 sets from (17b)), this def'n of the arithmetical hierarchy generalizes our previous one

$$\begin{array}{ccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & & \\ \Delta^0_1 & & \Delta^0_2 & & & \cdots & \\ & \Pi^0_1 & & \Pi^0_2 & & & \end{array}$$

Prop'n

① Σ^0_n is closed under bdd quant'n, $\exists$ quantification over $\mathbb{N}$, computable pre-image

② Π^0_n is closed under bdd quant'n, $\forall$ quant'n over $\mathbb{N}$, comp'e pre-image

③ Δ^0_n is closed under bdd quant'n, comp'e preimage

Pf: omitted.

A brief word on hyperarithmetical sets:

- we can define Σ^0_α for $\alpha < \omega_1^{CK}$
  least non-computable ordinal

by taking effective unions at limit stages, giving what is known as hyperarithmetical hierarchy.

$$\begin{array}{ccccccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & & \Sigma^0_\omega & & \Sigma^0_{\omega+1} & & \\ \Delta^0_1 & & \Delta^0_2 & & \cdots & \Delta^0_\omega & & \Delta^0_{\omega+1} & & \cdots \\ & \Pi^0_1 & & \Pi^0_2 & & & \Pi^0_\omega & & \Pi^0_{\omega+1} & & \end{array}$$

- as w/ arithmetic sets, can relativize notion of hyperarithmetical sets (i.e. sets A belonging to some Σ^0_α, Π^0_α for some α) to any oracle $x \in \mathcal{N}$

- then can prove the analogous connection with Borel sets (i.e. sets $A \in \underset{\sim}{\Sigma}^0_\alpha$ or $\underset{\sim}{\Pi}^0_\alpha$ for some α):

$$A \in \underset{\sim}{\Sigma}^0_\alpha / \underset{\sim}{\Pi}^0_\alpha / \underset{\sim}{\Delta}^0_\alpha \quad \text{(respectively)}$$

iff there is $x \in \mathcal{N}$ s.t.

$$A \in \Sigma^0_\alpha(x) / \Pi^0_\alpha(x) / \Delta^0_\alpha(x)$$
(respectively)

The analytical hierarchy

- We can extend the arithmetical hierarchy to the analytical hierarchy by taking projections along $\mathcal{N}$ instead of $\mathbb{N}$

<u>Def'n</u> We define the pointclasses Σ^1_n, Π^1_n, Δ^1_n for all spaces of the form $\mathbb{N}^k \times \mathcal{N}^l$, inductively as follows:

(i) A set A is Σ^1_1 iff it is the projection along $\mathcal{N}$ of a Π^0_1 set (equiv: an arithmetical set) i.e. if $B = B(n_1, \ldots, n_k, x_1, \ldots, x_\ell, x_{\ell+1})$ is a Π^c subset of $N^k \times \mathcal{N}^{\ell+1}$, then $A = \bigcup_{x \in \mathcal{N}} B_x$ is Σ^1_1, where $B_x = \{(n_1, \ldots, n_k, x_1, \ldots, x_\ell) : (n_1, \ldots, n_k, x_1, \ldots, x_\ell, x_{\ell+1}) \in B\}$

(2) $A \in \Pi^1_n$ iff A's complement is Σ^1_n

(3) $A \in \Sigma^1_{n+1}$ iff A is the projection along $\mathcal{N}$ of a Π^1_n set.

(4) $A \in \Delta^1_n$ iff A is both Σ^1_n and Π^1_n

A set in one of these pointclasses is called an ~~[illegible]~~ analytical set

(The corresponding boldface notions $\underset{\sim}{\Sigma}^1_n$, $\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$ are defined in exactly the same way, except in (1) we replace "arithmetical" with "Borel")

Terminology: "analytical set" does not mean the same thing as "analytic set" — the latter refers specifically to Σ^1_1 sets.

Prop'n

① Σ^1_n is closed under $\exists^{\mathbb{N}}$, $\forall^{\mathbb{N}}$, $\exists^{\mathbb{N}^{\mathbb{N}}}$ and computable pre-image

② Π^1_n is closed under $\exists^{\mathbb{N}}$, $\forall^{\mathbb{N}}$, $\forall^{\mathbb{N}^{\mathbb{N}}}$ and computable pre-image

③ Δ^1_n is closed under $\exists^{\mathbb{N}}$, $\forall^{\mathbb{N}}$, and comp'e pre-image.

Pf: omitted

- It follows quickly from the def'n that every arithmetical set is Δ^1_1
- Moreover can be shown that every hyperarithmetical set is Δ^1_1
- converse also true!

Theorem (Suslin-Kleene) A set is hyperarithmetical iff it is Δ^1_1

Pf: omitted

Thus the analytical hierarchy extends the (hyper) arithmetical hierarchy:

$$\begin{array}{cccccccccccc}
 & \Sigma^0_1 & & \Sigma^0_2 & & \Sigma^0_\alpha & & \Sigma^1_1 & & \Sigma^1_2 & \\
\Delta^0_1 & & \Delta^0_2 & \cdots & \Delta^0_\alpha & & \cdots & \Delta^1_1 & & \Delta^1_2 & \cdots \\
 & \Pi^0_1 & & \Pi^0_2 & & \Pi^0_\alpha & & \Pi^1_1 & & \Pi^1_2 &
\end{array}$$

As always, we can relativize notion of analytical set to an oracle $x \in \mathcal{N}$, and thus relate them to the corresponding boldface classes.

Prop'n Let Γ be one of $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ and let $\boldsymbol{\Gamma}$ be the corresponding boldface class ($\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ or $\boldsymbol{\Delta}^1_n$)
Given $A \subseteq \mathbb{N}^k \times \mathcal{N}^l$, TFAE:

(1). A is $\boldsymbol{\Gamma}$
(2) A is $\Gamma(x)$ for some $x \in \mathcal{N}$.

Pf: omitted.

Structural properties of the pointclass Π^1_1

Prop'n Let $X = \mathbb{N}^k \times \mathcal{N}^l$. There is a ~~[illegible]~~ universal Π^1_1 set for X.

Pf: - Let $U \subseteq \mathbb{N} \times \mathcal{N} \times X$ be a universal Π^0_1 set for X
- (we proved there is a universal Σ^0_1 set for X — take its complement to get a universal Π^0_1 set)

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- Then $\exists^{\mathcal{N}} u$ (i.e. projection along middle $\mathcal{N}$-coordinate of u), which is a subset of $\mathbb{N} \times X$, is a universal Σ^1_1 set for X
- its complement is universal Π^1_1 ✓

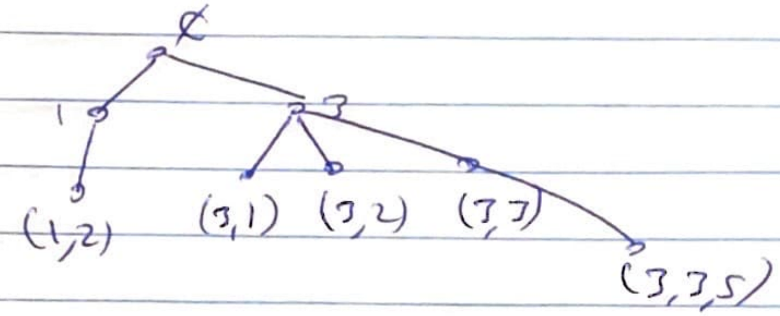
- We are now going to introduce an idea that has proved very fruitful: representing classes of countable structures as subsets of Baire space

- Can then locate the complexity (as a subset of $\mathcal{N}$) of a given class of structures in one of our hierarchies (i.e. figure out to which Σ^x_y, Π^x_y, Δ^x_y the class belongs)

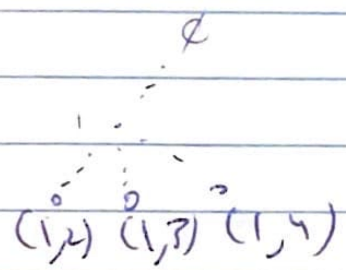
- First we define the classes we are going to represent

<u>Def'n</u> A subset $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is called a <u>tree</u> if it is closed under taking initial segments, i.e. if $s \in T$ then $s \restriction n \in T$ for every $n \leq \text{length}(s)$

E.g. $\{\emptyset, (1), (1,2), (3), (3,1), (3,2), (3,3), (3,3,5)\}$ is a tree



but $\{(1,2), (1,3), (1,4)\}$ is not



A tree T is <u>well-founded</u> if it has no infinite branches, i.e. there is no $y \in \mathbb{N}^{\mathbb{N}}$ s.t. $y \restriction n \in T$ for every $n \in \mathbb{N}$.

E.g. the tree above is well-founded (as is <u>any</u> finite tree)

Infinite trees can be well-founded too, e.g.