- a general open set $O \subseteq \mathcal{N}$ is any (arbitrary) union of basic open sets
$$O = \bigcup_{i \in I} N_{s_i}$$

↳ "arbitrary" a key word: effective DST interested in what happens when we only allow computable (or comp'ly enumerable) unions of N_s's.

- can also consider cartesian powers of $\mathcal{N}$, e.g. $\mathcal{N} \times \mathcal{N} = \mathcal{N}^2$, $\mathcal{N}^3$, etc. (akin to $\mathbb{R}^2, \mathbb{R}^3, \ldots$)

- Since we're gonna mix comp. theory w/ DST, will be useful to view $\mathcal{N}$ as a top. space too — but w/ discrete topology, i.e. all sets are open or equiv: basic open sets are singletons $\{n\}$

- we'll work w/ spaces of the form $X = \mathbb{N}^k \times \mathcal{N}^\ell$
- points in X are tuples $(n_1, \dots, n_k, u_1, \dots, u_\ell)$ where $n_1, \dots, n_k \in \mathbb{N}$, $u_1, \dots, u_\ell \in \mathcal{N}$

<u>Def'n</u> Let $X = \mathbb{N}^k \times \mathcal{N}^\ell$. A subset $A \subseteq X$ is Σ^0_1 (or <u>effectively open</u>) if there is a r.e. set $B \subseteq \mathbb{N}^k \times (\mathbb{N}^{<\mathbb{N}})^\ell$ s.t.

$$A = \bigcup_{(n_1, \dots, n_k, s_1, \dots, s_\ell) \in B} \{n_1\} \times \{n_2\} \times \dots \times \{n_k\} \times N_{s_1} \times \dots \times N_{s_\ell}$$

↳ e.g. if $X = \mathbb{N}$, then A is Σ^0_1 iff A is r.e. (i.e. we've generalized concept of Σ^0_1 from 117b) (notation matches!)

↳ OTOH if $X = \mathcal{N}$ then $A \in \Sigma^0_1$ iff there is a recursive $f: \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ whose outputs $f(n) = s_n$ determine basic open sets constituting A, i.e.

$$A = \bigcup_{n \in \mathbb{N}} N_{s_n} = \bigcup_{n \in \mathbb{N}} N_{f(n)}$$

A is Π^0_1 (or <u>effectively closed</u>) iff it is the complement of a Σ^0_1 set

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- Notice: since there are only ctbly many r.e. subsets $B \subseteq \mathbb{N}^k \times (\mathbb{N}^{<\mathbb{N}})^\ell$, there are only ctbly

many Σ^0_1 sets
$\Rightarrow \exists$ many Σ^0_1 (i.e. open sets) which are not $\Sigma^{\sim}_1$

- point is: we've focused our interest on open sets we can "construct" effectively out of basic open sets (using a computable f to enumerate)

Prop'n Let $X = \mathbb{N}^k \times \mathcal{N}^\ell$ and fix $A \subseteq X$. Then $A \in \Sigma^0_1$ iff there is a ~~[illegible]~~ r.e. relation $R \subseteq \mathbb{N}^k \times (\mathbb{N}^{<\mathbb{N}})^\ell$ s.t.

$$(n_1, \ldots, n_k, x_1, \ldots, x_\ell) \in X \text{ iff } \exists m \; R(n_1, \ldots, n_k, x_1 \restriction m, \ldots, x_\ell \restriction m)$$

(here: $x \restriction m$ - finite seq. of first m entries of x)

Pf. ($\Rightarrow$) - we'll do case when $X = \mathcal{N}$
- Sps $A = \bigcup_{s \in B} N_s$ for some r.e. $B \subseteq \mathbb{N}^{<\mathbb{N}}$
- then $B = \operatorname{im} f$ for some total computable $f: \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$
- but then:

$$x \in A \Leftrightarrow \exists s \in B \,(x \text{ extends } s) \;[\text{i.e. } x \in N_s]$$
$$\Leftrightarrow \exists n \,(x \text{ extends } f(n))$$

$\Leftrightarrow \exists m \exists n \; (f(n) = x \upharpoonright m)$

$\Leftrightarrow \exists m \; R(x \upharpoonright m)$

where $R(s) \Leftrightarrow \exists n \; (f(n) = s)$

which is a ~~[illegible]~~ r.e. relation ✓

($\Leftarrow$) immediate: sps $R \subseteq N^{<N}$ is r.e. s.t. $x \in A \Leftrightarrow \exists m \; R(x \upharpoonright m)$

Then

$$A = \bigcup_{s \in R} N_s \checkmark$$

- Prop'n says: $A \subseteq N$ is effectively open (Σ^0_1) iff there is r.e. R s.t. we can check $x \in A$ by checking if $x \upharpoonright m \in R$ for some truncation $x \upharpoonright m$ of x.

- the analog of a cts function (topological notion) in effective DST is a computable function (effective notion)
- More concretely: recall: f is cts if inverse image of every (basic) open set is open.
- in our context this gives e.g. $f: N \to N$ is cts iff $f^{-1}[N_s]$ is open $\forall s \in N^{<N}$, i.e.

$$f^{-1}[N_s] = \bigcup_{t \in B} N_t \quad \text{for some}$$

(not necessarily computable) $B \subset \mathbb{N}^{<\mathbb{N}}$, for every $s \in \mathbb{N}^{<\mathbb{N}}$

- Said another way: $\forall s \in \mathbb{N}^{<\mathbb{N}}$, $\{x \in \mathcal{N} : f(x) \in N_s\}$ is open
- or another way still: $\{(x,s) \in \mathcal{N} \times \mathbb{N}^{<\mathbb{N}} : f(x) \in N_s\}$ is open.

same def'n as prev. line, since here we view $\mathbb{N}^{<\mathbb{N}}$ as copy of $\mathbb{N}$ (discrete topology) so this set is open (in $\mathcal{N} \times \mathbb{N}^{<\mathbb{N}}$) iff for every $s \in \mathbb{N}^{<\mathbb{N}}$ the fiber $\{x : f(x) \in N_s\}$ is open (in $\mathcal{N}$)

- we're about to generalize notion of "computable function" in view of this def'n of cts.
- intuition for cts function (on $\mathcal{N}$) is: to determine specified initial segment of $f(x)$, need only know some initial segment of x
- int'n for computable function is: we have a "computation rule" for f s.t. to determine a specified initial part of f, need only know an initial part of x (to which we apply our rule)

Def'n Sps $X = \mathbb{N}^k \times \mathcal{N}^l$

① $f: X \to \mathbb{N}$ is computable if the relation $f(x) = n$ is effectively open i.e. the set $\{(x,n) \in X \times \mathbb{N} : f(x) = n\}$ is effectively open

② a function $f: X \to \mathcal{N}$ is computable if the relation $f(x) \in N_s$ is effectively open i.e. the set $\{(x,s) \in X \times \mathcal{N}^{<\mathbb{N}}$ s.t. $f(x) \in N_s\}$ is effectively open

implies $f^{-1}(B)$ is effectively open (in X) for all effectively open (i.e. r.e.) $B \subseteq \mathcal{N}$

implies

③ a function $f: X \to \mathbb{N}^m \times \mathcal{N}^n$ is effectively open if each of its coordinate w(s) projections is.

Ex's

① Defin $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n+1$ (here: $X = \mathbb{N}$). Then f is computable since the set $\{(n, \underset{f(n)}{n+1}) : n \in \mathbb{N}\}$ is effectively open in $\mathbb{N}^2$ (i.e. it's r.e.)

More generally: if $\mathrm{dom}(f) = \mathrm{ran}(f) = \mathbb{N}$, the def'n of "computable" above coincides w/ usual def'n of computable function $f: \mathbb{N} \to \mathbb{N}$

(recall: $f: \mathcal{N} \to \mathcal{N}$ is computable iff its graph $\{(x, f(x))\}$ is r.c.) ✓

② Define $f: \mathbb{N} \times \mathcal{N} \to \mathcal{N}$ (here $\mathcal{X} = \mathbb{N} \times \mathcal{N}$) by, if $x = (x_0, x_1, \ldots) \in \mathcal{N}$ and $n \in \mathbb{N}$, then $f(n, x) = nx :=$ $(n, x_0, x_1, \ldots)$

Then f is computable:

— need to check $O = \{(n, x, s) : \overset{f(n,x)}{\overbrace{nx}} \in N_s\}$ is eff'ly open, i.e. of form $\bigcup_{(n_1, s_1, s_2) \in B} \{n_1\} \times N_{s_1} \times \{s_2\}$ for some r.c. $B \subseteq \mathbb{N} \times \mathcal{N} \times \mathbb{N}^{<\mathbb{N}}$

— Let B = set of tuples (n, s_1, s_2) s.t. $s_2 = ns$. Then B not only r.c. but computable.

— not hard to check:

$O = \bigcup_B \{n\} \times N_s \times \{t\}$ for this B ✓

The classes ("point classes") of Σ^0_1 and Π^0_1 sets enjoy similar closure properties to the arithmetic classes they generalize.

<u>Prop'n</u> ① Σ^0_1 is closed under: finite unions and intersections, bounded quantification (over $\mathbb{N}$), projection along $\mathbb{N}$ (i.e. closure under $\exists n \in \mathbb{N}\ (\ldots)$)

projection along $\mathcal{N}$ (i.e. closure under $\exists x \in \mathcal{N} (\ldots)$), computable pre-image (i.e. preimage under a computable f)

② Π^0_1 is closed under: finite $\cup$'s and $\cap$'s; bdd quant'n; coprojection along $\mathbb{N}$ (i.e. closure under $(\forall n \in \mathbb{N})(\ldots)$) coprojection along $\mathcal{N}$ (i.e. closure under $\forall x \in \mathcal{N} (\ldots)$) computable pre-image

Pf. omitted ✓

We can still find universal Σ^0_1 sets in this larger context.

<u>Prop'n</u> Let $X = \mathbb{N}^k \times \mathcal{N}^d$. There is a <u>universal</u> Σ^0_1 set for X, i.e. a Σ^0_1 set $U \subseteq \mathbb{N} \times X$ s.t. for every Σ^0_1 subset $A \subseteq X$ there is $n \in \mathbb{N}$ s.t. $U_n = A$

<u>Pf.</u> - We do case when $X = \mathcal{N}$
- fix a computable enumeration $(W_e)_e$ of the r.e. subsets (of $\mathbb{N}^{<\mathbb{N}}$, in this case)
- Define $U(e, x) :\Leftrightarrow \exists m\ W_e(x \upharpoonright m)$

- Done, by prev. prop'n: explicitly given a Σ^0_1 $A \subseteq \mathcal{X}$, there is r.c. R s.t.

$x \in A \iff \exists m\, R(x, m)$

$\iff \uparrow M(e, x)$ where e is number of R in enumeration ✓

- Also as in previous context can relativize being Σ^0_1, Π^0_1 to an oracle $x \in \mathcal{N}$.

<u>Def'n</u> Let $X = \mathbb{N}^k \times \mathcal{N}^\ell$ and fix $x \in \mathcal{N}$. We say $A \subseteq X$ is $\Sigma^0_1(x)$ if there is a $\Sigma^0_1(x)$ subset $B \subseteq \mathbb{N}^k \times (\mathbb{N}^{<\mathbb{N}})^\ell$ s.t.

$$A = \bigcup_{(n_1, \dots, n_k, s_1, \dots, s_\ell) \in B} \{n_1\} \times \cdots \times \{n_k\} \times N_{s_1} \times \cdots \times N_{s_\ell}$$

↳ importance of relativizing like this is that it relates the effective notion of an open set to the classical one

<u>Prop'n</u> Fix $A \subseteq \mathbb{N}^k \times \mathcal{N}^\ell$.
TFAE:
① A is $\tilde{\Sigma}^0_1$ (i.e. <u>open</u> in $\mathbb{N}^k \times \mathcal{N}^\ell$)
② A is $\Sigma^0_1(x)$ for some $x \in \mathcal{N}$