- Sufficient for us: we're working wrt a fixed TM M and input w — and know length of comp'n of M on w.

<u>Claim</u>: $M_n(x,y,z)$ is definable by a poly-in-n length formula in lang of PrA

(En is similar — omit proof)

<u>Pf</u>: — we induct on n.

<u>For $n=0$</u>: — need to say "$x<2^{2^0} \wedge xy=z$" using only $<, S, +, 0$
- Can do it by:

$$(x=0 \wedge z=0) \vee (x=\overset{S(0)}{1} \wedge z=y) \checkmark$$

- <u>For $n=k+1$</u>: — Spse we've defined $M_k(x,y,z)$ by a poly-in-k length fmla; we define M_{k+1} by a poly-in-$k+1$ length fmla

- to write $xy=z$ using M_k when $x<2^{2^{k+1}}$, need to decompose x in terms of multiples $<2^{2^k}$
- if $x<2^{2^{k+1}}$ then $\sqrt{x}<2^{2^k}$, so $\lfloor\sqrt{x}\rfloor<2^{2^k}$ too
- Let $x_1 = \lfloor\sqrt{x}\rfloor$

- So $x = x_1^2 + w$ for some w
- Since $x < (x_1+1)^2$ we have
$x_1^2 + w < (x_1+1)^2 = x_1^2 + 2x_1 + 1$
- hence $w < 2x_1 + 1 \Rightarrow w \leq 2x_1$
- hence $x = x_1^2 + x_2 + x_3$ where both
$x_2, x_3 \leq x_1 < 2^{2k}$

Thus $xy = x_1(x_1 y) + x_2 y + x_3 y$

$\Rightarrow$ can define $M_{k+1}(x, y, z)$
(i.e. $x < 2^{2^{k+1}} \wedge xy = z$) by:

$$\exists x_1, x_2, x_3, u, w_1, w_2, w_3$$
$$[(x = x_1 + x_2 + x_3) \wedge (z = w_1 + w_2 + w_3)$$
$$\wedge (x_2 \leq x_1) \wedge (x_3 \leq x_1) \wedge$$
$$(x_1 < 2^{2^k}) \wedge (w_2 = x_2 y) \wedge (w_3 = x_3 y) \wedge (w_1 = x_1 u) \wedge (u = x_1 y)$$
$$M_k(x_2, y, w_2) \wedge M_k(x_3, y, w_3) \wedge M_k(x_1, u, w_1) \wedge M_k(x_1, y, u$$

- is defn of M_{k+1} in terms of M_k
- problem w: length of formula
is $\sim 4 \times$ (length of M_k - fmla)
- this would imply length of
fmlas M_k are exponentially growing
in k
- but we can write equiv fmla
using only one instance of M_k
— but need new vars r, s, t.

- just replace clauses subsequent to "$x < 2^{2^{k+1}}$" above with:

$\forall r,s,t$ (if (r,s,t) one of (x_2,y,w_2), (x_3,y,w_3), (x_1,u,w_1), (x_1,y,u) then $M_k(r,s,t)$)

really a 4-disjunction

- Length of this fmla is (length of formula defining $M_k(r,s,t)$) + C for some constant C independent of k

can get rid of $x < 2^{2^k}$ clause, since redundant w/ what follows

- this yields linear growth in length of M_k fmlas (by induction = length of M_0 + kC)

- notes I'm reading say you need a further mod to get linear growth, above only poly in k (I don't see it)
- but either way:
one instance of M_k to define M_{k+1} ⟹ fmla poly in k ✓
E_k can be handled similarly ✓

Some effective descriptive set theory

- For remainder of class: we'll explore analogies between computability theory and descriptive set theory (DST)
- First, recall from 117b the arithmetic hierarchy (of subsets of $\mathbb{N}$)

$$\begin{array}{ccccc} & \Sigma^0_1 & & \Sigma^0_2 & \\ \Delta^0_1 & & \Delta^0_2 & & \cdots \\ & \Pi^0_1 & & \Pi^0_2 & \end{array}$$

r.e. → Σ^0_1; recursive sets → Δ^0_1; co.r.e. → Π^0_1

- There is a similar looking hierarchy of sets on any topological space (e.g. $\mathbb{R}$)

$$\begin{array}{ccccc} & \underset{\sim}{\Sigma}^0_1 & & \underset{\sim}{\Sigma}^0_2 & \\ \underset{\sim}{\Delta}^0_1 & & \underset{\sim}{\Delta}^0_2 & & \cdots \\ & \underset{\sim}{\Pi}^0_1 & & \underset{\sim}{\Pi}^0_2 & \end{array}$$

open sets → $\underset{\sim}{\Sigma}^0_1$; clopen → $\underset{\sim}{\Delta}^0_1$; closed → $\underset{\sim}{\Pi}^0_1$

- Here, if X is our top. space, and $A \subseteq X$, then

- $A \in \underset{\sim}{\Sigma}^0_1$ iff A is open
  (i.e. Σ^0_1 is class of open sets)
- $A \in \underset{\sim}{\Pi}^0_n$ iff $A^c \in \Sigma^0_n$ (e.g. $A \in \Pi^0_1$ iff A closed)
- $\underset{\sim}{\Delta}^0_n = \underset{\sim}{\Sigma}^0_n \cap \Pi^0_n$
- $A \in \underset{\sim}{\Sigma}^0_{n+1}$ iff A is the union of ctbly many $\underset{\sim}{\Pi}^0_n$, i.e.
  $A = \bigcup_{k \in \mathbb{N}} B_k$ where $B_k \in \underset{\sim}{\Pi}^0_n$ $\forall k$

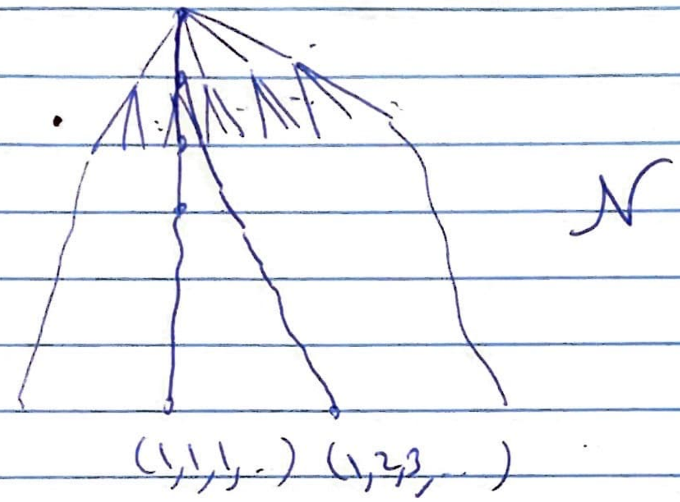
- For example, if $X = \mathbb{R}$ then $(0,1)$ and $\mathbb{R} \setminus \{0\}$ are $\underset{\sim}{\Sigma}^0_1$; $(-\infty, 0] \cup [1, \infty)$ and $\{0\}$ are $\underset{\sim}{\Pi}^0_1$; and $(0,1] = \bigcup_{k \in \mathbb{N}} [\frac{1}{k}, 1]$ is $\underset{\sim}{\Sigma}^0_2$

- Classical DST studies the Borel hierarchy of $\mathbb{R}$ (or really, $\mathcal{N}$, see below)

- Effective DST marries computability theory and classical DST:
  - studies hierarchy of effective Borel sets
  - are like Borel sets but in which we can only take computable unions when passing from one level to the next.

Effective open sets:

- top. space DST'ers work w/ $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ = space of infinite sequences of natural numbers.
  ↪ e.g. $(1,1,1,\dots)$ and $(1,2,3,\dots) \in \mathcal{N}$



- $\mathcal{N}$ is DST version of $\mathbb{R}$ — as a top. space $\mathcal{N} \cong$ irrationals (think: inf. seqs are like decimal expansions)
- Basic open sets in $\mathcal{N}$ are the sets $\mathcal{N}_s$, where $s = (s_0, s_1, \dots, s_k) \in \mathbb{N}^{<\mathbb{N}}$ is a finite sequence

$$\mathcal{N}_s := \{u \in \mathcal{N} : u = su'\}$$

↑ i.e. u begins w/ s (followed by some u')

- e.g. $(1,2,3,\dots) \in \mathcal{N}_{(1,2)}$