Semantics: first-order structures

Def'n given a language $L = \{R_i\} \cup \{f_j\} \cup \{c_k\}$
an L-structure (or just structure when L clear from context)
is an object of form

$$\mathcal{A} = \langle A, \{R_i^{\mathcal{A}}\}, \{f_j^{\mathcal{A}}\}, \{c_k^{\mathcal{A}}\} \rangle$$

where: - A is a nonempty set and
- each $R_i^{\mathcal{A}}$ is a relation on A of arity n_i (i.e. $R_i^{\mathcal{A}} \subseteq A^{n_i}$)
- each $f_j^{\mathcal{A}} : A^{m_j} \to A$ is a function on A of arity m_j
- each $c_k^{\mathcal{A}} \in A$ is an element of A.

"interp'y relations of the symbols R_i, f_j, c_k"

We often drop superscripts and confuse (e.g.) the actual relation $R^{\mathcal{A}}$ w/ the symbol R which represents it (notation abuse!)

Evaluation of terms: given a term $t(x_1, \ldots, x_n)$ we iteratively define a function $t^{\mathcal{A}} : A^n \to A$ as follows:

vars → among $x_1, \ldots, x_n$

- if $t(x_1, .., x_n)$ is x_i
  then $t^{\mathcal{A}}(a_1, .., a_n) = a_i$
- if $t(\cancel{x}_1, .., x_k)$ is c
  then $t^{\mathcal{A}}(a_1, .., a_n) = c^{\mathcal{A}}$
- if $t(x_1, .. x_n)$ is $f(t_1(\vec{x}), .., t_m(\vec{x}))$
  then $t^{\mathcal{A}}(\vec{a}) = f(t_1^{\mathcal{A}}(\vec{a}), .., t_m^{\mathcal{A}}(\vec{a}))$

e.g. if $L = \{<, S, +, \cdot, 0\}$ and
$\mathcal{N} = \langle \mathbb{N}, <^{\mathcal{N}}, S^{\mathcal{N}}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, 0^{\mathcal{N}} \rangle$ where
$<^{\mathcal{N}}$ is actual less than relation
$S^{\mathcal{N}}$ is actual successor, etc.

then if $t(x_1, x_2)$ is the term
$\underline{3}\, x_1^{\underline{2}} + \underline{2}\, x_2 + \underline{5}$ then $t^{\mathcal{N}}(10, 7) =$
$3 \cdot 10^2 + 2 \cdot 7 + 5 = 319$

Could let $\mathcal{M}$ be structure in same language everything the same except $+^{\mathcal{M}}$ is interpreted as exp function.
$n +^{\mathcal{M}} m = n^m$

then $t^{\mathcal{M}}(10, 7) = ((3 \cdot 10^2)^{2 \cdot 7})^5$

- Fix L and an L-structure $\mathcal{A}$
- We iteratively define when a fmla is true in $\mathcal{A}$ (w/ vars specified)

- Sps $\varphi(x_1, \ldots, x_k)$ a fmla
- Given $a_1, \ldots, a_k \in A$ (write $\vec{a}$) we define when φ <u>holds in $\mathcal{A}$</u> <u>w/ variable assignment</u> $x_i \to a_i$

(write: $\mathcal{A}, \bar{a} \models \varphi$)

(i) φ is $R_i(t_1, \ldots, t_m)$:

$$\mathcal{A}, \bar{a} \models \varphi \text{ iff } R_i^{\mathcal{A}}(t_1^{\mathcal{A}}(\vec{a}), \ldots, t_m^{\mathcal{A}}(\vec{a}))$$

(ii) φ is $t_1 = t_2$:

$$\mathcal{A}, \bar{a} \models \varphi \text{ iff } t_1^{\mathcal{A}}(\vec{a}) = t_2^{\mathcal{A}}(\vec{a})$$

(iii) φ is $\neg\chi$:

$$\mathcal{A}, \bar{a} \models \varphi \text{ iff } \mathcal{A}, \bar{a} \not\models \chi$$

(iv) φ is $\chi \wedge \tau$

$$\mathcal{A}, \bar{a}\ \varphi \models \text{ iff } \mathcal{A}, \bar{a} \models \chi \text{ and } \mathcal{A}, \bar{a} \models \tau$$

(v) φ is $\exists x_i\, \psi(x_1, \ldots, x_i, \ldots, x_k)$:

$\mathcal{A}, \bar{a} \models \varphi$ iff there is $b \in A$ s.t.

$$\mathcal{A}, (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \models \psi$$

We'll often write $\mathcal{A} \models \varphi(\bar{a})$ instead of $\mathcal{A}, \bar{a} \models \varphi$.

Ex: Consider $L = \{<, S, +, \cdot, 0\}$
Let $\mathcal{N} = \langle \mathbb{N}, <, S, +, \cdot, 0 \rangle$
(all symbols interpreted in usual way)

Then $\mathcal{N} \models$ "there are infinitely many primes"

i.e. $\mathcal{N} \models \forall x \exists y [x < y \wedge \underbrace{\overline{1 < y \wedge \forall z (\exists w (w \cdot z = y) \Rightarrow z = 1 \vee z = y}}_{\text{"y is prime"}}$

But $\mathcal{N} \not\models$ "4 is prime"
i.e. $\mathcal{N}, 4 \not\models$ "y is prime"

Consider sentence $\tau := \forall x \forall y (S(x) + y = S(x+y))$
Then $\mathcal{N} \models \tau$

Now let $\mathcal{M} = \langle \mathbb{N}, <, 10^{(\cdot)}, +, \cdot, 0 \rangle$
i.e. same as $\mathcal{N}$ except we interpret $S(\cdot)$ as $10^{(\cdot)}$ (so $S^{\mathcal{M}}(n) = 10^n$)

Then $\mathcal{M} \not\models \tau$ since $10^x + y \neq 10^{x+y}$ in general.

Def'n: - If φ a sentence in language of $\mathcal{A}$, read $\mathcal{A} \models \varphi$ as "$\mathcal{A}$ models φ"

- if Σ is a set of sentences (also called a theory) write

$\mathcal{A} \models \Sigma$ if $\mathcal{A} \models \varphi$ for every $\varphi \in \Sigma$.

<u>Ex</u>: Let $L = \{\cdot, e\}$ ($\cdot$ binary function, e constant)
Consider Σ consisting of following sentences:

(i) $\forall x \forall y \forall z \left((x \cdot y) \cdot z = x \cdot (y \cdot z) \right)$
(ii) $\forall x \left(x \cdot e = x \wedge e \cdot x = x \right)$
(iii) $\forall x \exists y \left(x \cdot y = e \wedge y \cdot x = e \right)$

Then given a structure $\mathcal{G} = \langle G, \cdot^{\mathcal{G}}, e^{\mathcal{G}} \rangle$ we have $\mathcal{G} \models \Sigma$ iff $\mathcal{G}$ is a group. (i.e. iff $\mathcal{G}$ has a assoc. operation $\cdot$ w/ identity e in which every el't has an inverse)

<u>Def'n</u> Given a theory Σ and a sentence φ we write $\Sigma \models \varphi$ if whenever $\mathcal{A} \models \Sigma$ we have $\mathcal{A} \models \varphi$ too

Say: Σ <u>logically implies</u> φ.

e.g. if φ is "$e \cdot e = e$" and Σ is as above then $\Sigma \models \varphi$ (since $e \cdot e = e$ is true in every group) a <u>tautology</u> is a sentence φ s.t. $\Sigma = \emptyset$ logically implies φ. (Write: $\models \varphi$)

Also say: a tautology φ is <u>logically valid</u>

Two sentences φ, χ are <u>logically equivalent</u> iff $\varphi \Leftrightarrow \chi$ is a tautology.

e.g. $c = c$

$\forall x \forall y\ R(x,y) \Leftrightarrow \forall y \forall x\ R(x,y)$

are tautologies
(hence $\forall x \forall y R(x,y)$ logically equiv. to $\forall y \forall x R(x,y)$)

Say a fmla $\varphi(x_1, \ldots, x_k)$ (poss. w/ free vars.) is a tautology iff $\forall x_1 \forall x_2 \ldots \forall x_k\ \varphi(x_1, \ldots, x_k)$ is.

- A useful way to measure complexity of a fmla: count # of $\forall \exists$ alternations @ beginning (think: arithmetic hierarchy)
- need a way of "pulling out" quantifiers for this to make sense in general.

Def'n A fmla τ is quantifier free if it contains no $\exists$'s or $\forall$'s.

A fmla is in prenex normal form if it is of the form

$$Q_1 y_1 \dots Q_n y_n \tau$$

where each Q_i is either $\forall$ or $\exists$ and τ is quantifier free.

e.g. $\forall x \exists y (x<y)$ is in p.n.f but $\forall x (0 < x \Rightarrow \exists y (S(y) = x))$ is not.

Prop'n every fmla is logically equiv. to a fmla in pnf.

Pf: idea: "keep pulling out quantifiers" only tricky part: sometimes need to replace vars before pulling out

e.g. $\exists x P(x) \wedge \exists x Q(x)$ equiv. to $\exists x \exists y (P(x) \wedge Q(y))$ but not $\exists x (P(x) \wedge Q(x))$ in general.

Following equivalences are all we need:

$\neg \exists x \varphi$ equiv to $\forall x \neg \varphi$

$\neg \forall x \varphi \iff \exists x \neg \varphi$

$\varphi \wedge \exists x \chi \iff \exists x(\varphi \wedge \chi)$

... as long as x not occurring in φ.

$\exists x \varphi \wedge \chi \iff \exists x(\varphi \wedge \chi)$

" "

... similar for $\forall x$ and $\vee$

- $Q_1 y_1 \dots Q_n y_n \chi(y_1, \dots, y_n, \dots) \iff Q_1 y_1' \dots Q_n y_n' \chi(y_1', \dots, y_n', \dots)$ where the y_i' different from y_j and also all vars in χ.

- If in $Q_1 y_1 \dots Q_n y_n \chi$ a var y appears more than once, can delete all instances of $Q_i y$ except rightmost.

Using these equivalences/transformations can iteratively pull out all quantifiers. Details of proof an exercise. Better to see example.

Ex. $(\exists x R(x,z) \wedge \neg \exists z S(z,w)) \wedge \exists x P(x,u)$

$(\exists x R(x,z) \wedge \forall z \neg S(z,w)) \wedge \exists x P(x,u)$

$\exists x (R(x,z) \wedge \forall z \neg S(z,w)) \wedge \exists x P(x,u)$

$\exists x \forall z' (R(x,z) \wedge \neg S(z',w)) \wedge \exists x P(x,u)$

$\exists x' \forall z' \left[\left((R(x',z) \wedge \neg S(z',w)) \right) \wedge \exists x P(x,u) \right]$

$\exists x' \forall z' \exists x \left[(R(x',z) \wedge \neg S(z',y) \wedge P(x,u)) \right]$

Given φ in pnf, we think of consecutive $\exists$ quantifiers as being in a single block, likewise for consecutive $\forall$'s

We count <u>alternations</u>. Say φ is $\exists_n$ if of form

$$\underbrace{\exists x_1^{(1)} \ldots \exists x_{k_1}^{(1)} \forall x_1^{(2)} \ldots \forall x_{k_2}^{(2)} \ldots}_{n \text{ blocks of quantifiers}} \quad \tau$$

Similar for $\forall_n$ but now begins with $\forall$ block

e.g. $\exists x \exists y \forall z \exists w \, \tau(x,y,z,w)$ is $\exists_3$

More generally: say φ is $\exists_n / \forall_n$ iff logically equiv. to a pnf ψ which is $\exists_n / \forall_n$.

# of quantifier alternations = useful measure of a fmla's complexity.

First-order definability

Fix a language L and L-structure $\mathcal{A} = \langle A, \{R_i^{\mathcal{A}}\}, \{f_j^{\mathcal{A}}\}, \{c_k^{\mathcal{A}}\} \rangle$

Some relations, functions, constants besides those labelled by R_i, f_j, c_k are "implicit" in $\mathcal{A}$.

Def'n We say $P \subseteq A^n$ is definable in $\mathcal{A}$ if there is a formula $\varphi(x_1, \ldots, x_n)$ s.t. $\vec{a} \in P \Leftrightarrow \mathcal{A} \models \varphi(\vec{a})$

a function $f : A^n \to A$ is definable if its graph $\mathrm{Graph}(f) \subseteq A^{n+1}$ is definable

a point $a \in A$ is definable iff $\{a\}$ is, i.e. iff there is $\varphi(x)$ s.t. $\mathcal{A} \models \varphi(b)$ iff $b = a$.

Def'n $P \subseteq A^n$ is definable with parameters if there is a formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in A$ s.t.

$$\vec{a} \in P \Leftrightarrow \mathcal{A} \models \varphi(\vec{a}, b_1, \ldots, b_m)$$

Similar for a function being definable w/ params

(Notice: every point of A is definable w/ params, namely by fmla $\varphi(x,b) := x = b$)

Ex's ① Sps $\mathcal{G} = \langle G, \cdot, e\rangle$ is a group. The inversion function $f : G \to G$, $f(x) = x^{-1}$

is definable by fmla $\varphi(x,y) := x \cdot y = e \wedge y \cdot x = e$

② The prime numbers $P \subseteq \mathbb{N}$ are definable in $\mathcal{N} = \langle \mathbb{N}, <, S, +, \cdot, 0\rangle$ by

$\varphi(x) :=$ "x is prime"

$:= (1 < x) \wedge (\forall y \forall z (y \cdot z = x) \Rightarrow y = 1 \vee y = x)$

③ Consider $\mathcal{R} = \langle \mathbb{R}, +, \cdot, 0, 1\rangle$ the ring of real numbers

Consider $\varphi(x,y) :=$

$\exists z (z \neq 0 \wedge \exists w (w \cdot w = z) \wedge x + z = y)$

then $\mathcal{R} \models \varphi(a,b)$ iff $a < b$ in $\mathbb{R}$

so $<$ is definable.