ε ε ε $\varepsilon>0$ fix $\varepsilon>0$ $x \in A$

(26)

else: $\alpha = \phi^*(\beta)$

Why: if k large enough s.t. $\forall i \leq n$ $\exists k_i < k$ s.t. $T(e, i, \bar{\beta}(k_i))$

exist by Kleene, since α total

then $\alpha \restriction n \subseteq \phi(\beta \restriction k)$

i.e. $b(\beta \restriction k) \geq n$

i.e. can compute init seg of α of length at least n by plugging in $\beta \restriction k$.

---

phi e

Corollary There is a sequence φ_e, $e \in \mathbb{N}$ of monotone recursive maps s.t.

$\alpha \leq_T \beta$ iff $\exists e$ s.t. $\varphi_e^*(\beta) = \alpha$

(follows from pf above)

Can use corollary to prove basic fact about $(D, \leq_T)$:

Theorem $\leq_T$ is not a linear order, i.e. $\exists \alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ s.t.

$\alpha \not\leq_T \beta$ and $\beta \not\leq_T \alpha$.

(21)

Pf "diagonalize against the φ_e"

- we build α, β inductively as limits of $s^n, t^n \in \mathbb{N}^{<\mathbb{N}}$
- $\forall n$ will have $|s^n| = |t^n| = \ell_n$ where $\ell_0 < \ell_1 < \dots$ strictly increasing in $\mathbb{N}$.
- and $\forall n$ $s^n \subseteq s^{n+1}$ $t^n \subseteq t^{n+1}$ (so $\alpha = \lim s^n$ $\beta = \lim t^n$ makes sense)

Idea: @ odd stages $n = 2e+1$: pick s^n, t^n so that $\varphi_e(t^n)$ differs from s^n in some place
$\Rightarrow$ ensures $\varphi_e^*(\beta) \neq \alpha$

@ even stages $n = 2e$: pick so $\varphi_e(s^n)$ differs from t^n in some place
$\Rightarrow \varphi_e^*(\alpha) \neq \beta$.

Let $s^{-1} = t^{-1} = \emptyset$ $\ell_{-1} = 0$

Now sps we have constructed s^{n-1}, t^{n-1} and ℓ_{n-1}

Case 1: $n = 2e+1$ is odd

Subcase (i): $\exists b \in \mathbb{N}^{\mathbb{N}}$ extending t^{n-1} s.t. $\varphi_e^*(b)$ is defined.

- define $t^n = b \upharpoonright \ell_n$ (so $t^n \supseteq t^{n-1}$) where we choose $\ell_n > \ell_{n-1}$ long enough so $|\varphi_e(t^n)| > |s^{n-1}|$ (possible by hypothesis)

- extend s^{n-1} to s^n of length ℓ_n in any way s.t. some new entry disagrees w/ corresponding entry in $\varphi_e(t^n)$

$\Rightarrow$ if α, β any extension of s^n, t^n will have $\varphi^*(\beta) \neq \alpha$.

Subcase (ii) $\not\exists$ such b.

- extend t^{n-1}, s^{n-1} to t^n, s^n of length $\ell_n > \ell_{n-1}$ arbitrarily
- for any β, α extending t^n, s^n will have $\varphi_e^*(\beta)$ undefined hence $\neq \alpha$.

Case 2: $n = 2e$

similar

More properties of $(D, \leq_T)$:

① Has a least element $\underset{\sim}{0} = [\alpha]_T$ where α is any recursive function

② Given $\underset{\sim}{a}, \underset{\sim}{b} \in D$, there is least upper bound $\underset{\sim}{c} \geq_T \underset{\sim}{a}, \underset{\sim}{b}$ where if $\alpha \in \underset{\sim}{a}$, $\beta \in \underset{\sim}{b}$ then $\underset{\sim}{c} = [\gamma]_T$ where $\gamma(n) = \langle \alpha(n), \beta(n) \rangle$

(can compute α, β from γ and if $\gamma' \geq \alpha, \beta$ then γ' computes γ)

③ given $\underset{\sim}{a} \in D$, $\{\underset{\sim}{b} : \underset{\sim}{b} \leq_T \underset{\sim}{a}\}$ is ctbl (immediate from corollary)

④ $|D| = |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$ "size continuum" (follows from corollary + basic cardinal arithmetic)

⑤ For any $\underset{\sim}{a} \in D$ there is $\beta \in 2^{\mathbb{N}}$ with $[\beta]_T = \underset{\sim}{a}$:

if $\underset{\sim}{a} = [\alpha]$ consider

$$A = \{\langle m, n \rangle \mid \alpha(m) = n\} \subseteq \mathbb{N}$$

then $\chi_A \equiv_T \alpha$

$\Rightarrow$

can work w/ subsets of $\mathbb{N}$ (i.e. el'ts of $2^{\mathbb{N}}$) instead of sequences

Another finer notion of reducibility

working w/ sets now instead of functions

Def'n: Given $A, B \subseteq \mathbb{N}$, say A reduces to B (or "many-to-one reduces") if there is a total recursive $f: \mathbb{N} \to \mathbb{N}$ s.t.

$$x \in A \iff f(x) \in B$$

Write: $A \leq_R B$

(equiv: $f^{-1}(B) = A$, equiv: $\chi_A = \chi_B \circ f$)

(f is called a reduction)

- to check membership in A, apply f and check membership in B
- "many-to-one" since f need not be injective
- observe: cannot have $x \in A$, $y \notin A$, $f(x) = f(y)$

↳ in this way reductions generalize char. functions

Notice: $A \leq_R B \Rightarrow A \leq_T B$ (immediate from $\chi_A = \chi_B \circ f$)

Converse not always true:

Complement A^c of A always $\leq_T A$ (so $A \equiv_T A^c$)

but if A is r.e. but not rec (e.g. A = halting problem) then if we had $A^c \leq_R A$ would have A^c r.e. (why?)

$\Rightarrow A$ rec, contradiction

Def'n Sps $\mathcal{C}$ is a family of subsets of $\mathbb{N}$. Say $B \subseteq \mathbb{N}$ is $\mathcal{C}$-complete if
① $B \in \mathcal{C}$
② if $A \in \mathcal{C}$ then $A \leq_R B$

RE(α) = r.e. in α subsets of $\mathbb{N}$

Theorem For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ there is $B \subseteq \mathbb{N}$ that is RE(α)-complete.
(In particular there is a RE-complete B)
regular r.e. sets

Pf: let $T(e, x, u)$ be p.r. set from prev. theorem w/ property that:
$A \in RE(\alpha)$ iff there is an $e \in \mathbb{N}$ s.t.
$x \in A \Leftrightarrow \exists \ell\, T(e, x, \bar{\alpha}(\ell))$

Now let $B = \{\langle m, n\rangle \mid \exists \ell\, T(m, n, \bar{\alpha}(\ell))\}$
clearly r.e. in α

Observe: if $A \in RE(\alpha)$ there is e_A s.t.
$A = \{n \mid \langle e_A, n\rangle \in B\}$ (i.e. $n \in A \Leftrightarrow \langle e_A, n\rangle \in B$)
Hence $f : \mathbb{N} \to \mathbb{N}$
$f(n) = \langle e_A, n\rangle$ gives reduction of A to B ✓

- can use theorem to define a canonical degree above a given $\underline{a} \in \mathcal{D}$

Def'n Given a Turing degree $\underset{\sim}{a} = [\alpha]_T$ define $\underset{\sim}{a}'$ (jump of $\underset{\sim}{a}$) to be $[B]_T$, where B is any RE(α)-complete set

note: if B' also RE(α)-complete then $B \equiv_R B'$ (hence $B \equiv_T B'$ hence $[B']_T = [B]_T$) so def'n makes sense

Claim $\underset{\sim}{a} <_T \underset{\sim}{a}'$

Pf: - suffices to check $B \not\leq_T \alpha$
- if $B \leq_T \alpha$ then since B RE(α)-complete, every RE(α) relation $\leq_T \alpha$.
- in particular, the universal relation $\omega(e,x) : \exists k T(e, x, \bar{\alpha}(k))$ is rec. in α

$\Rightarrow P(n) : \omega(n,n)$ is too
$\Rightarrow \neg P(n)$ too (rec. sets closed under comp)

$\Rightarrow$ for some e have
$$\neg P(n) \Leftrightarrow \exists k T(e, n, \bar{\alpha}(k))$$

but then $\neg P(e) \Leftrightarrow \exists k T(e, e, \bar{\alpha}(k))$

$$\Leftrightarrow \neg \exists k T(e, e, \bar{\alpha}(k))$$

contradiction

Fact: the jump is not a successor: e.g. the (unique) cttbl atomless Boolean algebra embeds in $\{\underset{\sim}{b} : \underset{\sim}{0} <_T \underset{\sim}{b} <_T \underset{\sim}{0}'\}$

The Arithmetic Hierarchy

"∃" Let Σ_1^0 = class of r.e. sets ↙ defined by $\exists k P(\vec{x}, k)$ P recursive

"∀" Π_1^0 = complements of r.e. sets ("co-r.e.")

$\Delta_1^0 = \Sigma_1^0 \cap \Pi_1^0$ = recursive sets

117a: $A \in R \Leftrightarrow A$ and $A^c \in RE$

Note: Σ_1^0 closed under ∃-quantification

hence Π_1^0 closed under ∀

Δ_1^0 closed under complements.

"∃∀" Continue: $\Sigma_2^0 = \{\exists k P(\vec{x}, k) \mid P \in \Pi_1^0\}$

"∀∃" $\Pi_2^0 = \{P^c \mid P \in \Sigma_2^0\}$

$\Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0$

In general: $\Sigma_{n+1}^0 = \{\exists k P(\vec{x}, k) \mid P \in \Pi_n^0\}$

$\Pi_{n+1}^0 = \{\neg P \mid P \in \Sigma_{n+1}^0\}$

$\Delta_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$

<u>Claim</u> $\Sigma^0_n \cup \Pi^0_n \subseteq \Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}$

<u>Pf</u>: induction (you try) ✓

The collection of sets appearing in this hierarchy are called <u>arithmetical sets</u> (aka arithmetical relations)

$$\text{Arith sets} = \bigcup_n \Sigma^0_n = \bigcup_n \Pi^0_n = \bigcup_n \Delta^0_n$$

<u>Prop'n</u> (1) Σ^0_n is closed under $\wedge, \vee$, bdd quantification, substitution by total recursive functions, and $\exists$-quantification

(2) Π^0_n is closed under " " " " " ", and $\forall$-quantification

(3) Δ^0_n is closed under " " " " " " " ", and $\neg$

<u>Pf</u>: Induction on n.

For n=1, did in 117a.

Assume for n.

We show (1), others similar

E.g. for $\wedge$:

Sps $R_1(\vec{x})$ is $\exists k P_1(k,\vec{x})$

$R_2(\vec{x})$ is $\exists l P_2(l,\vec{x})$

where $P_1, P_2 \in \Pi^0_n$

By induction $P(k,l,\vec{x}) = P_1(k,\vec{x}) \wedge P_2(l,\vec{x})$

is in Π^0_n

Hence $R_1 \wedge R_2 \in \Sigma^0_{n+1}$ since

$(R_1 \wedge R_2)(\vec{x})$ iff $\exists w P((w)_0, (w)_1, \vec{x})$

rest similar ✓

<u>Prop'n</u> Σ^0_n and Π^0_n have the enumeration property.

<u>Pf.</u> Induction on n. Knew for $n=1$ (117a)

Consider Σ^0_{n+1}.

For every fixed k want:

$W^{k+1}(e, x_1, \dots, x_k) \in \Sigma^0_{n+1}$ that is universal for k-ary Σ^0_{n+1} relations.

By induction can find in Π^0_n a relation