<u>Theorem</u> (Kleene's normal form theorem relative to α). There is a p.r. function $U: \mathbb{N} \to \mathbb{N}$ and a p.r. relation $T \subseteq \mathbb{N}^3$ s.t. for every oracle α, every $f: \mathbb{N}^n \to \mathbb{N}$ that is in $R(\alpha)$ has the form

$$U(\mu k[T(e, \langle x_1, \ldots, x_n \rangle, \bar{\alpha}(k))])$$

for some $e \in \mathbb{N}$

<u>Pf</u>: Recall (117a) there is a p.r. relation T' that is "universal for recursive(ly enumerable) relations"

i.e. for each rec(ly enumerable) $Q \subseteq \mathbb{N}^n$ there is $e \in \mathbb{N}$ s.t.

$$Q(x_1, \ldots, x_n) \text{ iff } \exists m T'(e, \langle x_1, \ldots, x_n \rangle, m)$$

(e codes the "computation tree" of Q)

<u>Now</u>: sps $f: \mathbb{N}^n \to \mathbb{N}$ is recursive in α

- by prev. thm there is p.r. P s.t.

$$f(x_1, \ldots, x_n) = y \text{ iff } \exists k P(x_1, \ldots, x_n, y, \bar{\alpha}(k))$$

Hence for some e we have

$f(\vec{x}) = y$ iff $\exists k \exists m\, T'(e, \langle \vec{x}, y, \bar{\alpha}(k) \rangle, m)$

iff $\exists s\, T'(e, \langle \vec{x}, y, \bar{\alpha}((s)_0) \rangle, (s)_1)$

$\langle k, m \rangle$

iff $\exists s\, T''(e, \langle \vec{x}, y, \bar{\alpha}(s) \rangle)$

where T'' is the pr. modification of T' that makes this line true

But then we can compute $f(\vec{x})$ as follows

$$f(\vec{x}) = (\mu k [T''(e, \langle \vec{x}, (k)_0, \bar{\alpha}((k)_1) \rangle)])_0$$

["search over codes for pairs $k = \langle y, s \rangle$ to find least one making $T''(e, \langle \vec{x}, y, \bar{\alpha}(s) \rangle)$ true and output y"]

Hence if we let $U(z) = (z)_0$ (is p.r.) and let T be the p.r. mod of T'' so that:

$$T(e, \langle \vec{x} \rangle, \bar{\alpha}(k)) \text{ iff } T''(e, \langle \vec{x}, (k)_0, \bar{\alpha}((k)_1) \rangle)$$

we get

$$f(\vec{x}) = U(\mu k [T(e, \langle \vec{x} \rangle, \bar{\alpha}(k))])$$

as desired. ✓

<u>Corollary</u> (Kleene's enumeration thm for $R(\alpha)$)
For every oracle $\alpha : \mathbb{N} \to \mathbb{N}$, $R(\alpha)$ has the enumeration property, i.e. there is a function $\varphi^\alpha : \mathbb{N}^2 \to \mathbb{N}$, recursive in α, s.t. for every unary f recursive in α, there is $e \in \mathbb{N}$ s.t. $\varphi^\alpha(e,x) = f(x)$ for all x (i.e. $\varphi^\alpha(e,\cdot) = f$)

<u>Pf</u>: Let $\varphi^\alpha(e,x) = U(\mu k\,[T(e, x, \bar{\alpha}(k))])$

should write $\langle x \rangle$

✓

<u>Def'n</u> a relation $P \subseteq \mathbb{N}^n$ is <u>recursive in α</u> iff χ_P is
~~...~~ is <u>recursively enumerable in α</u> iff there is $Q \subseteq \mathbb{N}^{n+1}$ recursive in α s.t.

$$P(\vec{x}) \text{ iff } \exists k\, Q(\vec{x}, k)$$

A rule of thumb: every thm about r/r.e. sets and functions in 117a has a "relativized to α" version, e.g.

<u>Thm</u> There is a p.r. relation $T(e,u,x)$ s.t. for every α and $P \subseteq \mathbb{N}$ r.e. in α there is $e \in \mathbb{N}$ s.t. $P(x)$ iff $\exists k\, T(e, x, \bar{\alpha}(k))$

Pf: you try.

Turing degrees

↳ we'd like to understand how much "extra" info a non-recursive oracle α gives us, and which oracles give us more or less.

Def'n Spse $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ are oracles. Say α is *Turing reducible* to β and write $\alpha \leq_T \beta$ if $\alpha \in R(\beta)$

Claim for all oracles α, β, γ

(1) $\alpha \leq_T \beta$ iff $R(\alpha) \subseteq R(\beta)$
(2) $\alpha \leq_T \alpha$
(3) $\alpha \leq_T \beta \leq_T \gamma \Rightarrow \alpha \leq_T \gamma$

Pf: immediate.

"Turing equivalent"

Write $\alpha \equiv_T \beta$ iff $\alpha \leq_T \beta$ and $\beta \leq_T \alpha$

<u>Observe</u> $\equiv_T$ is equiv. relation on N^N
($=$ set of all $\alpha : N \to N$)

Write $[\alpha]_T$ for equiv. class of α

"Turing degree" of α

Write D for $\{[\alpha]_T : \alpha \in N^N\}$

on D, $\leq_T$ is a partial order

on N^N, $\leq_T$ is a "quasi-order" since can have $\alpha \leq_T \beta \leq_T \alpha$ but $\alpha \neq \beta$.

will sometimes use notation $\underset{\sim}{a}, \underset{\sim}{b}$ etc to denote classes $[\alpha]_T, [\beta]_T$, etc. in D.

so $\underset{\sim}{a} = [\alpha]_T$ for any $\alpha \in \underset{\sim}{a}$

Sequence notation

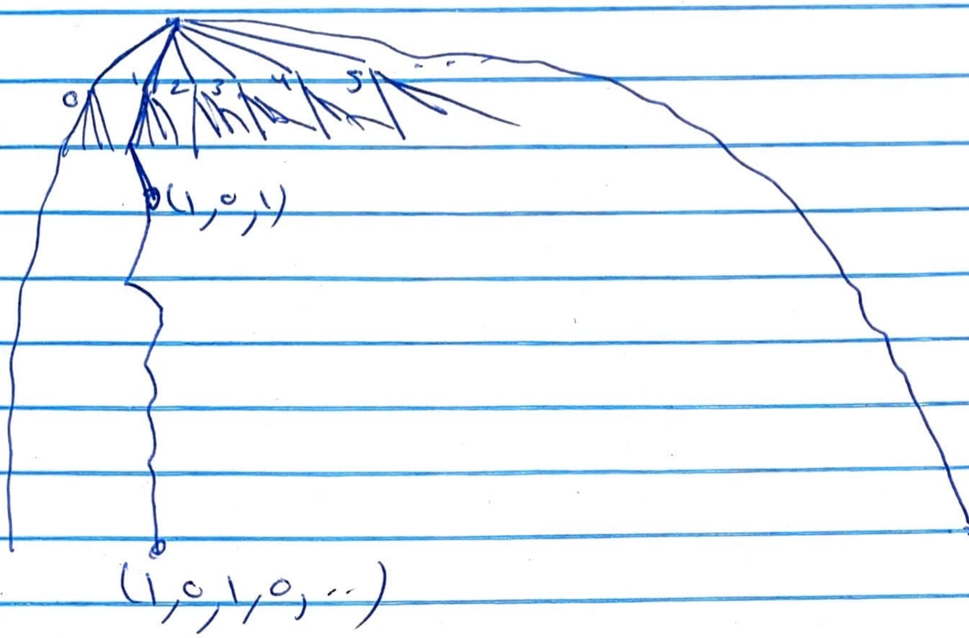
- write $\mathbb{N}^n$ for set of sequences $s = (s_0, \dots, s_{n-1})$ of length n
- $\mathbb{N}^{<\mathbb{N}}$ is $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ = set of all finite sequences on $\mathbb{N}$. (includes $\emptyset$ - empty sequence)
- $\mathbb{N}^{\mathbb{N}}$ = all inf sequences $\alpha = (a_0, a_1, \dots)$ = all oracles
- write $|s|$ for length of s
  $= n$ if $s \in \mathbb{N}^n$
  $= \infty$ $s \in \mathbb{N}^{\mathbb{N}}$

  ↳ not same as "length(x)" which returns length of sequence coded by $x \in \mathbb{N}$.
- if $|s| \geq n$ write $s \restriction n$ for first n entries of s,
  e.g. $(1,0,1,2,3) \restriction 3 = (1,0,1)$
  $(1,1,1,\dots) \restriction 3 = (1,1,1)$
- if $|t| \geq |s|$ say t extends s (write $s \subseteq t$) if $t \restriction |s| = s$

  s is "initial segment" of t

  e.g. $(1,1,2,2,3,3)$ extends $(1,1,2)$
- if a is finite sequence and b is finite or infinite write ab for a concatenated w/ b
  e.g. $(1,1)(2,3,4,\dots) = (1,1,2,3,4,\dots)$

Pic of $\mathbb{N}^{\mathbb{N}}$:



$\mathbb{N}^{<\mathbb{N}}$ = finite approx'ns of points in $\mathbb{N}^{\mathbb{N}}$.

Another def'n of $\le_T$:

- a total function $\phi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is monotone if $s \subseteq t \Rightarrow \phi(s) \subseteq \phi(t)$

- e.g. if $f: \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ is any function can define $\phi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by:
$$\phi(a_0, \dots, a_{n-1}) = f(a_0)f(a_1)\dots f(a_{n-1})$$

↳ then ϕ is monotone

- e.g. if $f(n) = \underbrace{(n, n, \dots, n)}_{n \text{ times}}$

then ϕ so defined is monotone and e.g.

$\phi(2,0,3) = f(2)f(0)f(3)$
$= (2,2)\phi(3,3,3) = (2,2,3,3,3)$

(not all monotone ϕ of this form)

- if ϕ is monotone can define
  $\phi^*: N^N \to N^N$ by
  $\phi^*(u) = \lim_{n\to\infty} \phi(u \restriction n) = \bigcup_{n\in N} \phi(u\restriction n)$

- def'n makes sense by monotonicity
  (e.g. for ϕ above, $\phi^*(0,1,2,3,\ldots)$
  $= (1,2,2,3,3,3,\ldots)$)

- we consider $\phi^*(u)$ defined only for
  $u \in N^N$ where $|\phi(u\restriction n)| \to \infty$
  ↳ so ϕ^* can be partial
  (e.g. $\phi^*(1,2,0,0,0,\ldots)$ undef for ϕ^* above)

- say that a map $\phi: N^{<N} \to N^{<N}$ is
  recursive if the map $\hat{\phi}: N \to N$
  $\hat{\phi}(\langle s\rangle) = \langle \phi(s)\rangle$
  is recursive.
  (notice: $\hat{\phi}$ is partial - only defined on codes!)

  (need to work in terms of codes - don't have def'n for "recursive" for functions on $N^{<N}$)

nother way of viewing $\leq_T$

Theorem Sps $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$.
Then $\alpha \leq_T \beta$ iff there is recursive monotone $\phi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ s.t. $\phi^*(\beta) = \alpha$

Pf ($\Leftarrow$) Sps ϕ is recursive, mon. w/ $\phi^*(\beta) = \alpha$.
- How to compute $\alpha(n)$?
→ given: $\forall$ large enough K ← so that $|\phi(\beta\restriction K)| \geq n+1$
$(\alpha(0), \ldots, \alpha(n)) = \alpha \restriction n+1$ is $\subseteq \phi(\beta \restriction K)$

so:
- working w/ codes and $\bar{\phi}$ we have:
$$\alpha(n) = \left(\bar{\phi}\left(\bar{\beta}\left(\mu K[\text{length}(\bar{\beta}(K)) > n]\right)\right)\right)_n$$

shows: α recursive in $\bar{\beta}$ (and hence in β)
i.e. $\alpha \leq_T \beta$.

($\Rightarrow$) Want: a monotone (hence total) $\phi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ s.t. ϕ(init seg of β) = init seg of α.

→ use $\alpha \leq_T \beta$ — By Kleene $\exists$ p.r. functions U, T and $e \in \mathbb{N}$ s.t.
$$\alpha(n) = U(\mu K[T(e, n, \bar{\beta}(K))])$$

↳ to compute $\alpha(n)$, need to know long enough init seg of β to find min K making $T(e, n, \bar{\beta}(K))$ true

- idea w/ long enough seg, can compute all of $\alpha(0), \ldots, \alpha(n)$
- need to define ϕ for all $s \in \mathbb{N}^{<\mathbb{N}}$ not just init segs of β

First attempt:

output up seg of length $|s|$

$$\phi(s) = \big(U(\mu k[T(e,n,\langle s\restriction k\rangle)])\big)_{0 \le n < |s|}$$

but in general: s may not be long enough to compute all entries n up to $|s|$

so we bound: let $b(s) = \max m < |s|$ s.t. [redacted] $\forall n \le m$ there is $k < |s|$ s.t. $T(e,n,\langle s\restriction k\rangle)$

"Clearly" b is p.r. (all searches bounded in s)

Now define $\phi(s) = \big(U(\mu k[T(e,n,\langle s\restriction k\rangle)])\big)_{0 \le n \le b(s)}$

ϕ is:

- monotone (for longer s, still searching for least witness)
- recursive

output is seg of length $b(s)$
- could be $b(s) = 0$ $\Rightarrow$ empty output!

else: $\alpha = \phi^*(\beta)$.

Why: if k large enough s.t. $\forall i \leq n$ $\exists k_i < k$ s.t. $T(e, i, \bar{\beta}(k_i))$

exist by Kleene, since α total

then $\alpha \upharpoonright n \subseteq \phi(\beta \upharpoonright k)$

i.e. $b(\beta \upharpoonright k) \geq n$

i.e. can compute init seg of α of length at least n by plugging in $\beta \upharpoonright k$.

Corollary There is a sequence φ_e, $e \in \mathbb{N}$ of monotone recursive maps s.t.

phi

e

$$\alpha \leq_T \beta \text{ iff } \exists e \text{ s.t. } \varphi_e(\beta) = \alpha.$$

(follows from pf above)

Can use corollary to prove basic fact about $(D, \leq_T)$:

Theorem $\leq_T$ is not a linear order, i.e. $\exists \alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ s.t.

$$\alpha \not\leq_T \beta \text{ and } \beta \not\leq_T \alpha.$$