By the deduction lemma and using that Q_0 is <u>finite</u> we have:

$\sigma \in T'$ iff $Q_0 \cup T \vdash \sigma$

iff $T \vdash (\bigwedge Q_0 \Rightarrow \sigma)$

conjunction of sentences in Q_0

iff $(\bigwedge Q_0 \Rightarrow \sigma) \in T$

Since the map $\langle \sigma \rangle \mapsto \langle \bigwedge Q_0 \Rightarrow \sigma \rangle$ is recursive, it follows T' is recursive (to check if $\sigma \in T'$, compute $\bigwedge Q_0 \Rightarrow \sigma$ and check if in T, which is rec.)

→ contradicts our prev. theorem: any consistent theory extending Q_0 is undecidable (since it reps every rec. f/R)

(Moral proof: if T decidable, if I add a single sentence Q to form T', then still decidable, since any proof of σ from T' can be rewritten as a proof of $Q \Rightarrow \sigma$ from T'. But if $Q = \bigwedge Q_0$, T' <u>not</u> decidable)

<u>Def'n</u> Sps $\mathcal{A}$ is a structure in a language $L_{\mathcal{A}}$, and $\mathcal{B}$ is a structure in $L_{\mathcal{B}}$. We say that $\mathcal{A}$ is <u>definable in $\mathcal{B}$</u>, if:

① A (universe of $\mathcal{A}$) is definable as a subset of B, by a formula φ (written in $L_{\mathcal{B}}$) possibly w/ parameters.
i.e. $\varphi = \varphi(x, x_1, \ldots, x_n)$ and for some $b_1, \ldots, b_n \in B$
$A = \{a \in B : \mathcal{B} \models \varphi(a, b_1, \ldots, b_n)\}$

② Each $R^{\mathcal{A}}$ is definable (possibly w/ params) in $\mathcal{B}$

③ Graph of each $f^{\mathcal{A}}$ is definable (w/ params) in $\mathcal{B}$

(note: each constant $c^{\mathcal{A}}$ definable by $x = c^{\mathcal{A}}$ since allowing params).

<u>Theorem</u> Sps $\mathcal{A}$ is an $L_{\mathcal{A}}$ structure definable in some $L_{\mathcal{B}}$-structure $\mathcal{B}$.
If $\mathcal{A}$ is strongly undecidable and $L_{\mathcal{A}}$ is finite then $\mathcal{B}$ is strongly undecidable.

Pf:- We show we can reduce problem to when $\mathcal{A}$ is definable w/o parameters
- need to pass to an enriched structure $\mathcal{B}^*$.

- Since $L_{\mathcal{A}}$ is finite, there is a finite subset $P \subseteq B$ containing all parameters needed to define A as well as each $R^{\mathcal{A}}$, $f^{\mathcal{A}}$.

- Let $L_{\mathcal{B}^*} = L_{\mathcal{B}} \cup \{c_b : b \in P\}$.
- we've introduced a new constant symbol for every $b \in P$.
- Let $\mathcal{B}^*$ = struct w/ universe B (= universe of $\mathcal{B}$) in which all R, f, c symbols from $L_{\mathcal{B}}$ interpreted same as in $\mathcal{B}$ and each new c_b interpreted as b (el't it's intended to label).

- Observe that $\mathcal{A}$ is definable in $\mathcal{B}^*$ w/o params, by some formulas defining it in $\mathcal{B}$ but w/ new constant symbols substituted for parameters they stand for.

<u>Claim</u> if $\mathfrak{B}^*$ is strongly undecidable, so is $\mathfrak{B}$.

<u>Pf</u>: - Spse T is theory (written in L_β) and $\mathfrak{B} \models T$. WTS: T is undecidable
- Consider L_{β^*} theory T^* defined by:

$$T^* = \{\sigma^* : T \vdash \sigma^*\}$$
$$= \{\varphi(c_{b_1}, \ldots, c_{b_k}) : \varphi(x_1, \ldots, x_k) \text{ is fmla in } L_\beta \text{ s.t. } T \vdash \varphi(c_{b_1}, \ldots, c_{b_k})\}$$

- Then T^* is L_{β^*} theory w/ $\mathfrak{B}^* \models T^*$ (since $\mathfrak{B} \models T$ we have $\mathfrak{B}^* \models T$ and hence $\mathfrak{B}^* \models$ any fmla deducible from T)

- Hence T^* is undecidable by hypothesis.

- Now <u>notice</u>: since the formulas in T do not contain any of the constants c_b, any proof of a formula $\varphi(c_{b_1}, \ldots, c_{b_k})$ from T can only introduce the new constant symbols via the substitution deduction axiom:

$$\forall x\, \varphi(x) \Rightarrow \varphi(x | c_b)$$

every free instance of x replaced w/ c_b

or the equality axiom

$$c_b = c_b$$

(but this is actually an instance of substitution since $\forall x (x = x)$ is an axiom)

- If instead at any step in which we introduce a new constant c_{b_i} we instead introduced a (substitutable) free variable x_i and o/w kept proof same we would prove instead $\varphi(x_1, .., x_k)$ from T

- Thus: $\varphi(c_{b_1}, .., c_{b_k}) \in T^*$ iff
  $T \vdash \varphi(c_{b_1}, .., c_{b_k})$ iff
  $T \vdash \varphi(x_1, .., x_k)$ iff
  $T \vdash \forall x_1, .., \forall x_k \, (\varphi(x_1, .., x_k))$ iff
  $\forall x_1, .. x_k \, \varphi(x_1, .., x_k) \in T$

(moral: if we can establish a statement about c_b, some proof must actually establish statement for every x)

But then, since $\langle \varphi(\vec{c}) \rangle \to \langle \forall \vec{x} \varphi(\vec{x}) \rangle$ is recursive, if T were computable then T^* would also be. Hence T is not computable, i.e. undecidable ✓

(Note: some details here swept under rug concerning going from $\varphi(\vec{c})$ to $\forall \vec{x}\, \varphi(\vec{x})$ recursively, since we need to make sure replacing vars are substitutable..)

So: WLOG may assume A is definable in B w/o params (can pass to B^*, show it is strongly undecidable then B is too..)

But then: we get a natural interpretation $L_A \to \pi (L_B, Th(B))$:

take $\pi_\forall$ to be fmla defining $A \subseteq B$
π_R " " R^A in B
π_f " " f^A in B
π_c " " c^A in B

(This actually gives an interpretation since our base theory is $Th(B)$, which not only proves but contains ~~the~~ the statements "$\exists x\, \pi_\forall(x)$" "$\pi_f$ is the graph of a function" "π_c defines a singleton in A" since these statements are true in B by virtue of the fact that they define A in B)

As before, for every formula σ written in L_A a translated formula σ_π written in L_B s.t.

$$A \models \sigma \text{ iff } B \models \sigma_\pi$$

Since L_A is finite the translation map $\langle\sigma\rangle \to \langle\sigma_\pi\rangle$ is recursive

- Sps now that T_B is some theory s.t. $B \models T_B$.
- Define $T_A = \{\sigma : \sigma_\pi \in T_B\}$

  written in L_A — in L_B

- Then $A \models T_A$, since T_A asserts some things about A in its own lang. that $\{\sigma_\pi : \sigma \in T_A\}$ asserts about A as a definable subset of B

$\Rightarrow T_A$ is undecidable by hypothesis

If T_B were decidable then since $\langle\sigma\rangle \to \langle\sigma_\pi\rangle$ is recursive would have T_A is too

$\Rightarrow T_B$ is undecidable ✓

- Can use theorem to "translate" undecidability of $\mathcal{N} = \langle \mathbb{N}, <, S, +, \cdot, 0 \rangle$ to other "less arithmetic" classes of structures.

- E.g. since $\mathcal{N}$ is (easily) definable in $\mathcal{Z} = \langle \mathbb{Z}, +, *, 0, 1 \rangle$ (ring of integers) it follows theory of rings is undecidable!

- i.e. Let T be theory consisting of following statements (theory of rings):
  - "the universe is an abelian group under + w/ identity 0"
  - "$*$ is associative"
  - $\forall x \, (x * 1 = 1 * x = x)$
  - $\forall a, b, c \, (a * (b + c) = a * b + a * c$
    $\wedge \; (b + c) * a = b * a + c * a)$

Then since $\mathcal{Z} \models T$ and $\mathcal{Z}$ is strongly undecidable, T is undecidable, i.e. we cannot decide recursively in general if a statement written in lang of rings is true in all rings (i.e. provable)