<u>Theorem</u> (Gödel's 2nd Incompleteness theorem)

Sps Q satisfies HB conditions, represents every recursive function, and ω consistent

Then $Q \nvdash Con(Q)$

<u>Pf</u>: Let σ_Q be Gödel sentence of Q, so that

$$Q \vdash \neg\sigma_Q \Leftrightarrow Prvbl_Q(\langle \underline{\sigma_Q} \rangle) \quad (*)$$

Then by ①

$$Q \vdash Prvbl_Q(\underline{\neg\sigma_Q \Leftrightarrow Prvbl_Q \langle \sigma_Q \rangle}) \quad (**)$$

By HB condition (2)

$$Q \vdash Prvbl_Q(\langle \underline{\sigma_Q} \rangle) \Rightarrow Prvbl_Q(\underline{Prvbl_Q(\langle \sigma_Q \rangle)})$$

So, using ③ and $\Leftarrow$ direction of $*$ we get

$$Q \vdash Prvbl_Q(\langle \underline{\sigma_Q} \rangle) \Rightarrow Prvbl_Q(\langle \underline{\neg\sigma_Q} \rangle)$$

Now, using $\Rightarrow$ direction of $(*)$ we get

$$Q \vdash \neg\sigma_Q \Rightarrow (Prvbl_Q(\langle \underline{\sigma_Q} \rangle) \wedge Prvbl_Q(\langle \underline{\neg\sigma_Q} \rangle))$$

So by our lemma $Q \vdash \neg\sigma_Q \Rightarrow \neg Con(Q)$

i.e. $Q \vdash Con(Q) \Rightarrow \sigma_Q$

– Hence if $Q \vdash Con(Q)$
would have $Q \vdash \sigma_Q$
but we knew $Q \nvdash \sigma_Q$
Hence $Q \nvdash Con(Q)$ ✓

– So question is: which Q's satisfy HB conditions?

– Turns out: Q_0 is not strong enough!

(Q_0 surprisingly weak in other ways: e.g. $Q_0 \vdash \underline{n} + \underline{m} = \underline{m} + \underline{n}$ for every $m, n \in \mathbb{N}$, but $Q_0 \nvdash$ "+ is commutative" (i.e. $\forall x \forall y \; x + y = y + x$)
↳ it follows: there are nonstandard models $\mathcal{A} \models Q_0$ in which + is not commutative for some nonstandard elts in $\mathcal{A}$)

(Despite this, $Q_0 \nvdash Con(Q_0)$; but there are closely related systems that do prove their own consistency)

Def'n Peano arithmetic (PA) is the system obtained by adding all induction axioms to PA; i.e. for every formula $\varphi(x_1,\dots,x_n,x)$ we add the fmla:

$$\forall x_1,\dots,x_n \left[\left(\varphi(\vec{x},\underline{0}) \wedge (\forall z\, \varphi(\vec{x},z) \Rightarrow \varphi(\vec{x},z+1))\right) \Rightarrow \forall z\, \varphi(\vec{x},z) \right]$$

Note: PA is (clearly) a recursive set of axioms, since one can computably recognize if a given fmla is an induction axiom

Note: PA is an infinite set of axioms whereas Q_0 was finite.
Can be shown: PA is not finitely axiomatizable.

Theorem PA satisfies the HB provability conditions. Hence (assuming PA is consistent)

$$PA \nvdash Con(PA)$$

The same is true for any consistent recursive extension of PA.

Proof: Omitted (it's long!)

Remark: it follows that $PA \cup \{\neg Con(PA)\}$ is consistent! Therefore (by completeness theorem) has a model $\mathcal{A}$. What does this mean?

$\neg Con(PA)$ asserts "there is an x coding a proof of $\perp$"

does not imply that for some $n \in \mathbb{N}$, "$\underline{n}$ codes a proof of $\perp$" as then n would code an actual proof of $\perp$ from PA

rather, in $\mathcal{A}$, there is a nonstandard number N ($>$ all $\underline{n}^{\mathcal{A}}$) that "codes" a "proof" of $\perp$.

Some culture

An objection to incompleteness theorems: σ_φ, $Con(PA)$ etc. aren't real "mathematical" statements but just coded "metamathematical" ones.

Not hard to find simple, natural statements independent of Q_0, e.g. commutativity of $+$ — but this just reflects not having access to induction

What about PA? Are there "natural" "mathematical" σ indep over PA?

Ramsey Theorems

Given $n \in \mathbb{N}$ and a set S,
$[S]^n :=$ set of n-element subsets of S ("unordered n-tuples")

For $a, b, n, k \in \mathbb{N}$ we write
$$b \to (a)^n_k$$
to denote statement:
for every $c : [b]^n \to k$
there is $A \subseteq b$ with $|A| \geq a$
s.t. $c \restriction [A]^n$ is constant

(Here, we identify $b = \{0, 1, \ldots, b-1\}$ w/ set of its predecessors)

In words: if we color (using c) the n-element subsets of $b = \{0, 1, \ldots, b-1\}$ w/ k-many colors, there is $A \subseteq b$ w/ $|A| \geq a$ s.t. $[A]^n$ is colored by a single color.

Ramsey's Theorem states that for every a, n, k there is b s.t. $b \to (a)^n_k$

<u>Fact</u>: PA $\vdash$ Ramsey's Theorem

Now, we write

$$b \to^* (a)^n_k$$

for same statement as above, except we insist $|A| > \min A$

Ramsey's * Theorem says: for every a, n, k there is b s.t. $b \to^* (a)^n_k$

<u>Theorem</u> Ramsey* Theorem is true

<u>Theorem</u> (Paris - Harrington) ⊗

$$PA \nvdash \text{Ramsey's* Theorem}$$

(So Ramsey* independent from PA)

<u>Goodstein Sequences</u>:

Given $n, k \in \mathbb{N}$, $k > 1$, the <u>pure base k representation</u> of n is obtained as follows:

- write base k expansion of n

- write the exponents of k in their base k expansions
- then the exponents of exponents etc. until we reach 0 in all exponents

E.g. the pure base 2 expansion of 8 is:

$$8 = 2^3 = 2^{2^1+2^0} = 2^{2^{2^0}+2^0}$$

of 19:

$$19 = 2^4 + 2^1 + 2^0$$
$$= 2^{2^2} + 2^{2^0} + 2^0$$
$$= 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

The Goodstein Sequence $G(n,k)$ is defined inductively:

$G(n,1) = n$

$G(n,k+1)$ obtained by writing $G(n,k)$ in pure base $k+1$, changing every $k+1$ to $k+2$, then subtracting 1 from entire sum.

E.g. $G(19,1) = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$

$G(19,2) = 3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$

$= 3^{3^{3^{3^0}}} + 3^{3^0} \approx 7 \times 10^{13}$

$G(19,3) = 4^{4^{4^4}} + 4^{4^4} - 1 \approx 10^{154}$

$G(19,4) \approx 10^{2184}$

...

Theorem For every n, there is K with $G(n,K) = 0$.

(hence $G(n,N) = 0$ for $N \geq K$ too)

Pf: tricks w/ ordinals ✓

Theorem (Paris - Kirby) PA $\nvdash$ Goodstein's Theorem.

✓

Strong undecidability

Recall: a collection of sentences T is undecidable if $\langle Conseq(T) \rangle$ is not recursive

We call T a theory if $T = Conseq(T)$ (i.e. T is closed under $\vdash$)

A theory T is finitely axiomatizable if there is a finite $S \subseteq T$ s.t. $Conseq(S) = T$

- We define a concept of undecidability for structures
- Sps L is a language and $\mathcal{A}$ is an L-structure.

Def'n $\mathcal{A}$ is strongly undecidable if for every theory T w/ $\mathcal{A} \models T$ we have T is undecidable.

Observe: strong undecidability implies
① $Th(\mathcal{A})$ is undecidable, and (on the other end of the spectrum):
② $Taut(L)$ (i.e. collection of valid formulas, provable from logical axioms alone) is undecidable

Theorem $\mathcal{N} = \langle \mathbb{N}, <, s, +, \cdot, 0 \rangle$ is strongly undecidable.

Pf. Let T be a theory written in L_{ar} with $\mathcal{N} \models T$ and sps T is decidable.

Consider the theory $T' = Conseq(T \cup Q_0)$