Pf: Sps first $Q \vdash \sigma_Q$.
Then there is $k \in \mathbb{N}$ s.t. $\mathrm{Proof}_Q(\langle \sigma_Q \rangle, k)$

By representability of Proof_Q we have:
$Q \vdash \mathrm{Proof}_Q(\underline{\langle \sigma_Q \rangle}, \underline{k})$

Hence $Q \vdash \exists y\, \mathrm{Proof}_Q(\underline{\langle \sigma_Q \rangle}, y)$

i.e. $Q \vdash \mathrm{Prvbl}_Q(\underline{\langle \sigma_Q \rangle})$

cTCH we get $Q \vdash \neg \mathrm{Prvbl}_Q(\underline{\langle \sigma_Q \rangle})$
from $(**)$, contradicting consistency. ✓

Now sps $Q \vdash \neg \sigma_Q$

Then by a deduction from $(**)$
$Q \vdash \mathrm{Prvbl}_Q(\underline{\langle \sigma_Q \rangle})$ i.e.
$Q \vdash \exists y\, \mathrm{Proof}_Q(\underline{\langle \sigma_Q \rangle}, y)$

By ω-consistency and rep'ability of Proof_Q, there is $p \in \mathbb{N}$ s.t.
$Q \vdash \mathrm{Proof}_Q(\underline{\langle \sigma_Q \rangle}, \underline{p})$

But Proof_Q here represents the real Proof_Q relation on $\mathbb{N}$ hence $\mathrm{Proof}_Q(\langle \sigma_Q \rangle, p)$ really holds, i.e.

p codes a real proof of σ_Q
i.e. $Q \vdash \sigma_Q$, contradiction.

Next time: how to get rid of hypothesis of ω-consistency.

Rosser's Trick:

- want: σ that is independent of Q w/out assuming ω-consistency of Q.
- Rosser found one
- need to work in a slightly more specific situation:
  ① we assume Q written L_{ar} (so we have access to $<$)
  ② Assume $Q_0 \subseteq \text{Conseq}(Q)$ (so we can use lemmas abt $<$ proved from Q)

Define: $\text{Proof}^{\neg}_Q(x, y)$ to be relation:

$$\text{Proof}_Q(y) \wedge (y)_{\text{length}(y)-1} = \text{Neg}(x)$$

where $\text{Neg}(x) = \begin{cases} \langle \neg \varphi \rangle & \text{if } x = \langle \varphi \rangle \\ 0 & \text{o.w.} \end{cases}$

$\text{Proof}^{\neg}_Q(x,y)$ asserts "y is a code for a proof of $\wedge$ x's negation"
fmla coded by

- $\mathrm{Proof}^{\neg}_{Q}$ is recursive (since $\mathrm{Neg}(x)$ is clearly)
- Hence representable in Q, say by a fmla we also write as $\mathrm{Proof}^{\neg}_{Q}(x,y)$

Now consider:

$$\varphi(x) := \forall y\,(\mathrm{Proof}_Q(x,y) \Rightarrow \exists y' \leq y\,(\mathrm{Proof}^{\neg}_{Q}(x,y'$$

"For every code y for a proof of x there is a smaller code y' of $\mathrm{Neg}(x)$"

- Apply the fixed point lemma to this $\varphi(x)$
- We get a formula ρ_Q s.t.

(*) $$Q \vdash \rho_Q \Leftrightarrow \forall y\,(\mathrm{Proof}_Q(\underline{\langle \rho_Q \rangle}, y) \Rightarrow \exists y' \leq y\ \mathrm{Proof}^{\neg}_{Q}(\underline{\langle \rho_Q \rangle}, y')$$

<u>Theorem</u> (Rosser) Sps Q is collection of sentences written in L_{ar} and $Q_0 \subseteq \mathrm{Conseq}(Q)$ (so in particular: Q reps every recursive function). Assume Q is consistent and recursive.

Then $Q \nvdash \rho_Q$ and $Q \nvdash \neg\rho_Q$

<u>Pf</u>: Sps first $Q \vdash \rho_Q$

let k be a code for a proof

Then $\mathrm{Proof}_Q(\langle \rho_Q \rangle, k)$ and hence by rep-ability we have:

$$Q \vdash \mathrm{Proof}_Q(\langle \underline{\rho_Q} \rangle, \underline{k})$$

By (a deduction from) (*) we have

$$Q \vdash \exists y' \le \underline{k}\,(\mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, y'))$$

<u>Recall</u>: we had a lemma that showed (for every $k \in \mathbb{N}$)

$$Q_0 \vdash \forall x(x \le \underline{k} \Rightarrow x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{k})$$

Hence $Q \vdash$ " "

since $\mathrm{Conseq}(Q) \supseteq Q_0$.

Hence we can deduce

(3) $$Q \vdash \mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, \underline{0}) \vee \ldots \vee \mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, \underline{k})$$

It follows: there is actual $l \le k$ s.t. $Q \vdash \mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, \underline{l})$

<u>Why</u>: For every $l \le k$ we have either $\mathrm{Proof}^{\neg}_Q(\langle \rho_Q \rangle, l)$ or $\neg\mathrm{Proof}^{\neg}_Q(\langle \rho_Q \rangle, l)$ (and exactly one, by consistency)

$\Rightarrow$ by rep'ability $Q \vdash$ exactly one of $\mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, \underline{l})$

$\neg\mathrm{Proof}^{\neg}_Q(\langle \underline{\rho_Q} \rangle, \underline{l})$

If for every $l \leq k$, $Q \vdash \neg(\ldots)$
get a contradiction w/ (ξ) above.

So for some $l \leq k$ $Q \vdash \mathrm{Proof}^{\neg}_{Q}(\underline{\langle \rho_Q \rangle}, \underline{l})$

But then we really have $\mathrm{Proof}^{\neg}_{Q}(\langle \rho_Q \rangle, l)$
i.e. l codes an actual proof of $\neg \rho_Q$
from Q

$\Rightarrow Q \vdash \neg \rho_Q$, contradiction

Now spse $Q \vdash \neg \rho_Q$
Let n be a code for a proof of $\neg \rho_Q$.
i.e. $\mathrm{Proof}^{\neg}_{Q}(\langle \rho_Q \rangle, n)$

But then $Q \vdash \mathrm{Proof}^{\neg}_{Q}(\underline{\langle \rho_Q \rangle}, \underline{n})$
by representability

OTOH, by (*), since $Q \vdash \neg \rho_Q$ we
deduce $Q \vdash \exists y\, (\mathrm{Proof}_Q(\underline{\langle \rho_Q \rangle}, y) \wedge$
$\forall y' \leq y\, (\neg \mathrm{Proof}^{\neg}_{Q}(\underline{\langle \rho_Q \rangle}, y'))$

Combining w/ prev line gives
$Q \vdash \exists y \leq n\, (\ldots)$

But then, similarly to above

there is some actual $m \in \mathbb{N}$ s.t.

$$Q \vdash \mathrm{Proof}_Q(\langle \underline{\rho_Q} \rangle, \underline{m})$$

so that m really codes a proof of ρ_Q and so

$$Q \vdash \rho_Q$$

contradiction ✓

The 2nd Incompleteness Theorem

↳ informally, says any sufficiently strong axiom system Q for arithmetic cannot prove its own consistency.

Observation to get started: we showed if Q consistent then $Q \nvdash \sigma_Q$, where σ_Q is Gödel sentence (this is direction where we didn't need ω-consistency)

i.e. we proved the "sentence"

$$Q \text{ consistent} \Rightarrow \neg \mathrm{Prvbl}(\langle \underline{\sigma_Q} \rangle)$$

Sps we could formalize this statement (write in L_{ar})

and then

formalize its proof in Q, i.e. show
$Q \vdash$ "Q is consistent" $\Rightarrow \neg \mathrm{Prvbl}_Q(\langle \sigma_Q \rangle)$

Then if $Q \vdash$ "Q is consistent", since we know $Q \vdash \neg \mathrm{Prvbl}(\langle \sigma_Q \rangle) \Leftrightarrow \sigma_Q$
we could deduce $Q \vdash \sigma_Q$
contradicting $Q \nvdash \sigma_Q$.
$\Rightarrow$ $Q \nvdash$ "Q is consistent"

How to formalize this "proof"?
Need to assume more about Q.

<u>Def'n</u> Q satisfies the <u>Hilbert-Bernays provability conditions</u> if for all fmlas φ, ψ:

① If $Q \vdash \varphi$, then $Q \vdash \mathrm{Prvbl}_Q(\langle \varphi \rangle)$
② $Q \vdash \mathrm{Prvbl}_Q(\langle \varphi \rangle) \Rightarrow \mathrm{Prvbl}_Q(\langle \mathrm{Prvbl}_Q(\langle \varphi \rangle) \rangle)$
③ $Q \vdash (\mathrm{Prvbl}_Q(\langle \varphi \rangle) \wedge \mathrm{Prvbl}_Q(\langle \varphi \Rightarrow \psi \rangle)) \Rightarrow \mathrm{Prvbl}_Q(\langle \psi \rangle)$

- We write "Q is consistent" as follows:
let $\bot$ denote $\exists x (x \neq x)$
let Con(Q) denote
$\neg \mathrm{Prvbl}_Q(\langle \bot \rangle)$

We'll need:

Lemma Sps Q satisfies the HB conditions. Then for any φ we have:

$$Q \vdash Con(Q) \Leftrightarrow (\neg Prvbl_Q(\underline{<\varphi>})) \vee (\neg Prvbl_Q(\underline{<\neg\varphi>}))$$

Pf: - $\forall x\,(x=x)$ is a logical axiom
- we can deduce $\neg\neg\forall x(x=x)$ from this
  i.e. $\neg\exists x(x\neq x)$
  i.e. $\neg\bot$
- So $Q \vdash \neg\bot$
- Hence by HB(i)
  $Q \vdash Prvbl_Q(\underline{<\neg\bot>})$

(†) Thus: $Q \vdash \neg Con(Q) \Rightarrow (Prvbl_Q(\underline{<\bot>}) \wedge Prvbl_Q(\underline{<\neg\bot>}))$

(Left conjunct is immediately deducible from $\neg Con(Q)$, right is above)

By HB(i), applied to an axiom we have:

(Γ) $Q \vdash Prvbl_Q(\underline{<\bot \Rightarrow (\neg\bot \Rightarrow \varphi)>})$

Applying HB(3) twice we get:

$$Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\varphi} \rangle)$$

(Why: $Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\bot} \rangle)$
combining w/ (Γ) and using ③
we get $Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\neg\bot \Rightarrow \varphi} \rangle)$
but also $Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\neg\bot} \rangle)$
Hence applying ③ again $Q \vdash \neg \mathrm{Con}(Q)$
$\Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\varphi} \rangle)$)

By a similar arg we get

$$Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\neg\varphi} \rangle)$$

Combining:

$$Q \vdash \neg \mathrm{Con}(Q) \Rightarrow \mathrm{Prvbl}_Q(\langle \underline{\varphi} \rangle) \wedge \mathrm{Prvbl}_Q(\langle \underline{\neg\varphi} \rangle)$$

which gives backward direction $\Leftarrow$ of claim

Conversely, we have

$$Q \vdash \mathrm{Prvbl}_Q(\langle \underline{\varphi \Rightarrow (\neg\varphi \Rightarrow \bot)} \rangle)$$

Applying ③ again (twice) we get

$$Q \vdash \mathrm{Prvbl}_Q(\langle \underline{\varphi} \rangle) \wedge \mathrm{Prvbl}_Q(\langle \underline{\neg\varphi} \rangle) \Rightarrow \neg \mathrm{Con}(Q)$$

which is $\Rightarrow$ direction ✓