We knew $Q_0 \vdash \underline{l_i} \neq \underline{0}$ by Lemma (4)
and $Q_0 \vdash \forall w(w < \underline{m} \Rightarrow w = \underline{0} \vee w = \underline{1} \vee \dots \vee w = \underline{m-1})$

by Lemma (1)
and $Q_0 \vdash \psi(\underline{\vec{c}}, \underline{i}, \underline{l_i})$ for each $i < m$

(again since ψ reps g)
So indeed $Q_0 \vdash \forall w(w < \underline{m} \Rightarrow \exists z(z \neq 0 \wedge \psi(\underline{\vec{c}}, v, z)))$ ✓

<u>Def'n</u> A set of sentences Q is <u>undecidable</u> if $\langle \mathrm{Conseq}(Q) \rangle$ is not recursive.

<u>Theorem</u> (Gödel's 1st Incomp. thm – abstract existence form)
Suppose Q is a recursive set of axioms in the language of arithmetic, and $Q_0 \subseteq Q$. If Q is consistent, then Q is incomplete and undecidable

<u>Pf</u>: Since Q_0 represents every recursive function/relation, so does Q. By theorem preceding previous one, this $\Rightarrow$ $\langle \mathrm{Conseq}(Q) \rangle$ is not

Q-representable. Hence $\langle \text{Conseq}(Q) \rangle$ is not recursive (i.e. Conseq(Q) is not recursive). We showed Conseq(Q) is always r.e. when Q is recursive, and also: if Q is complete then Conseq(Q) actually recursive. Hence Conseq(Q) is incomplete ✓

- this proof is nice: uses complexity theory we've developed
- later: we give another pf in which we produce an explicit σ s.t. $Q \nvdash \sigma$ and $Q \nvdash \neg\sigma$

We've considered axiom systems in $L_{ar} = \{<, +, \cdot, S, 0\}$. We can extend incompleteness theorem to systems in other languages using <u>interpretations</u>

<u>Def'n</u> Sps L and L^* are two languages and T is a set of L-formulas.

An interpretation of (L, T) in L^* (write $L^* \rhd (L, T)$) is a collection of the following:

① An L-formula π_u (u for universe!) s.t. $T \models \exists x\, \pi_u(x)$

② assignments
$$R(\vec{x}) \to \pi_R(\vec{x})$$
$$f(\vec{x}) \to \pi_f(\vec{x}, y)$$
$$c \to \pi_c(x)$$

From the relation, function, constant symbols R, f, c in L^* to L-formulas π_R, π_f, π_c of correct arity.

such that:

(a) $T \models \forall x_1 \ldots \forall x_n \, (\pi_u(x_1) \wedge \cdots \wedge \pi_u(x_n) \Rightarrow \underset{\text{unique}}{\exists!}\, y\, (\pi_u(y) \wedge \pi_f(x_1, \ldots, x_n, y)))$

for each function symbol $f \in L^*$

(b) $T \models \exists!\, x\, (\pi_u(x) \wedge \pi_c(x))$ for each constant symbol c in L^*

<u>The point</u>: if $A \models T$ (in long L) then A "contains" an L^*-structure namely let

$$A^\pi = \{ b \in A : A \models \pi_u(b) \} \text{ nonempty by ①}$$

then we get a full L^* structure $A^\pi = \langle A^\pi, \{R_i^{A^\pi}\}, \{f_j^{A^\pi}\}, \{c_k^{A^\pi}\} \rangle$ by interpreting each R as

$$\{(a_1, \ldots, a_k) \in (A^\pi)^k : A \models \varphi_R(a_1, \ldots, a_k)\}$$

and etc. for f, c.

Example: Sps T is the full theory of $(\mathbb{Z}, \cdot, +)$ (in the lang $L = \{\cdot, +\}$)

We can get a natural interpretation of $L_{ar} = \{<, S, +, \cdot, 0\}$ in which the universe is exactly $\mathbb{N}$, using Lagrange's theorem (every $n \in \mathbb{N}$ is the sum of four squares):

$\pi_{\forall}(x) := \exists x_1, x_2 x_3, x_4 \left(x = x_1^2 + x_2^2 + x_3^2 + x_4^2 \right)$
$\pi_{<}(x, y) := (x \neq y) \wedge (\exists z (\pi_{\forall}(z) \wedge x + z = y))$
$\pi_{S}(x, y) := \pi_{<}(x, y) \wedge \forall z (\pi_{<}(x, z) \Rightarrow y = z \vee \pi_{<}(y, z))$

$\pi_{+}(x, y, z) := x + y = z$
$\pi_{\cdot}(x, y, z) := x \cdot y = z$
$\pi_{0}(x) := x + x = x$

It follows from the basic first-order properties of $(\mathbb{Z}, +, \cdot)$ that this is an interpretation, i.e. $T \vDash$ ① + ② a.b. for these assignments.

Then: from the model $\mathcal{Z} = (\mathbb{Z}, +, \cdot)$ (which is def a model of $T = Th(\mathcal{Z})$) we get the structure $\mathcal{Z}^{\pi}$ in the language of arithmetic and (by Lagrange) $\mathbb{Z}^{\pi}$ is exactly the standard model of arithmetic $\langle \mathbb{N}, <, S, +, \cdot \rangle$

↳ in this sense, we can "interpret" $\mathcal{N}$ in the ring of integers $\mathbb{Z}$.

- Once we've interpreted every L^* relation, function and constant symbol, we can interpret every L^* fmla.
- e.g. we can interpret an atomic fmla such as $\varphi := R(f(x), y, c)$ by the fmla

$$\exists z \exists w \left(\Pi_f(x,z) \wedge \Pi_c(w) \wedge \Pi_R(z,y,w) \right),$$

(write: φ_π for this fmla)

and likewise for other atomic fmlas

- Once atomic fmlas interpreted, inductively define:

$$(\neg \varphi)_\pi := \neg \varphi_\pi$$
$$(\varphi \Rightarrow \psi)_\pi := \varphi_\pi \Rightarrow \psi_\pi$$
$$(\exists x \varphi)_\pi := \exists x \left(\Pi_U(x) \wedge \varphi_\pi(x) \right)$$

- Now: if Q a collection of L^* sentences, an interpretation of Q in (L, T) is an interp. of L^* in (L, T) s.t.

For each $\sigma \in Q$ we have $T \models \sigma_\pi$

(equiv: $T \vdash \sigma_\pi$, by completeness)

<u>Prop'n</u> (1st Incompleteness theorem, general form)

Sps that T is a consistent, recursive collection of sentences in a language L and there is an interpretation of Q_0 in (L, T) s.t. the map $\langle\sigma\rangle \to \langle\sigma_\pi\rangle$ is recursive.

minor technical condition

(Think: T at least as strong as Q_0)

<u>Then</u>: T is undecidable and incomplete.

<u>Pf</u>: As before suffices to prove T is undecidable (i.e. Conseq(T) non-rec.)

- Sps toward contrad. Conseq(T) is recursive.
- Let $Q = \{\varphi : \varphi_\pi \in \text{Conseq}(T)\}$
- Since Conseq(T) is recursive, so is Q (to check if $\psi \in Q$, compute ψ_π (recursive) and check if $\psi_\pi \in$ Conseq(T) (recursive))
- Since Conseq(T) closed under deductions so is Q (i.e. $Q = \text{Conseq}(Q)$)

(<u>Why</u>: Any proof of $Q \vdash \sigma$ translates to a proof Conseq(T) $\vdash \sigma_\pi$)

($\Rightarrow \sigma_\pi \in Q$ too)

So: $\mathrm{Conseq}(Q) = Q$ is recursive (i.e. Q decidable)

Likewise Q is consistent, since T is (any proof of a contradiction could be translated)

So: Q is recursive, consistent, decidable and $Q \supseteq Q_0$, contradicting Gödel. ✓

<u>Gödel's approach</u>: an explicit indep. sentence.

<u>Idea</u>: Find σ that asserts "σ is not provable"

<u>More explicitly</u>: want a fmla that says "the fmla w/ Gödel code n_0 is not provable" — whose Gödel code is n_0!

If $K(n)$ is Gödel code of the formula "the formula w/ Gödel code n is not provable" need a lemma to assure there is n with $K(n) = n$.

Sps $L \supseteq \{0, S\}$ and Q is a collection of L-sentences that represents every recursive function/relation.

<u>Lemma</u> (Gödel fixed point lemma)
For every L-formula $\varphi(x)$ w/ a single free variable, there is a sentence σ s.t.

$$Q \vdash \sigma \Leftrightarrow \varphi(\underline{\langle\sigma\rangle})$$

(Think: $\varphi(n)$ asserts "fmla coded by n is not provable")

<u>Pf</u>: Consider function:

$$Sub(n,m) = \begin{cases} \langle\varphi(\underline{m})\rangle & \text{if } n \text{ codes a fmla } \varphi(x) \\ 0 & \text{o.w.} \end{cases}$$

- As before, take for granted Sub is recursive.
- Let $\chi(x,y,z)$ be fmla representing $Sub(n,m)=z$ in Q

Define $\psi(x)$ to be the formula:

$$\exists z (\chi(x,x,z) \wedge \varphi(z))$$

(read as: "$\varphi(Sub(x,x))$")

- Let $n_0 = \langle \psi(x) \rangle$
- We claim $\sigma := \psi(n_0)$ works
  $(= \psi(\langle \psi(x) \rangle))$
- Observe: $\langle \sigma \rangle = \langle \psi(n_0) \rangle = Sub(n_0, n_0)$

(☆) So: $Q \vdash \varphi(\underline{\langle \sigma \rangle}) \Leftrightarrow \varphi(\underline{Sub(n_0, n_0)})$

(trivial, these are same, just substituting

Write m for $Sub(n_0, n_0)$.
Since $Sub(n_0, n_0) = m$ and χ reps Sub in Q
we have $Q \vdash \chi(\underline{n_0}, \underline{n_0}, \underline{m})$

Hence $Q \vdash \varphi(\underline{m}) \Leftrightarrow (\chi(\underline{n_0}, \underline{n_0}, \underline{m}) \wedge \varphi(\underline{m}))$

again trivial (knowing $Q \vdash \chi(\underline{n_0}, \underline{n_0}, \underline{m})$)

Hence $Q \vdash \varphi(\underline{m}) \Leftrightarrow \exists z(\chi(\underline{n_0}, \underline{n_0}, z) \wedge \varphi(z))$
($\Leftarrow$ allowed since $Q \vdash$ uniqueness of outputs of χ)

i.e. $Q \vdash \varphi(\underline{Sub(n_0, n_0)}) \Leftrightarrow \underbrace{\exists z(\chi(\underline{n_0}, \underline{n_0}, z) \wedge \varphi(z))}_{\text{but this is } \sigma}$

So ~~considering~~ combining with (*) we are done ✓

Define a relation
$\text{Proof}_Q(x,y)$
iff y codes a proof of x from Q, i.e.
$\text{Proof}_Q(y) \wedge (y)_{\text{length}(y)-1} = x$

Then Proof_Q is recursive, and hence rep'able in Q, say by a formula that we also denote $\text{Proof}_Q(x,y)$ (notation abuse)

Define a function $\text{Prvbl}_Q(x)$ iff $\exists y\, \text{Proof}_Q(x,y)$

Let $\varphi(x)$ be the formula $\neg\text{Prvbl}_Q(x)$.

If we apply fixed point lemma to this we get a ~~s~~ sentence σ_Q s.t.

(**) $\quad Q \vdash \sigma_Q \Leftrightarrow \neg\text{Prvbl}(\underline{\langle\sigma_Q\rangle})$

We call σ_Q the Gödel sentence of Q.

<u>WTS</u>: σ_Q is independent from Q
i.e. $Q \nvdash \sigma_Q$ and $Q \nvdash \neg\sigma_Q$.

<u>Turns out</u>: need to make a technical assumption about Q for this to be possible.

Assume Q is ω-consistent, i.e. for each fmla $\psi(y)$, if $Q \vdash \exists y\, \psi(y)$ then there is $p \in \mathbb{N}$ s.t. $Q \nvdash \neg\psi(p)$
(i.e. if Q proves there is a witness to ψ, it doesn't prove there is no standard witness)

(Fact: Robinson arithmetic, Peano, etc. ω-consistent in ZFC)

Notice that if ψ represents a relation R, then $Q \nvdash \neg\psi(p) \Rightarrow Q \vdash \psi(p)$

Here is Gödel's original formulation:
<u>Theorem</u> (1st incomp. thm, o.g. form)
Assume that Q is recursive, ω-consistent, and represents every recursive function/relation.
Then $Q \nvdash \sigma_Q$ and $Q \nvdash \neg\sigma_Q$