(Rewriting this page)

Theorem Sps L contains the symbols $0, S$ and let Q be a consistent set of sentences in this language. If every recursive function and relation is representable in Q, then $\langle Conseq(Q) \rangle$ is not rep'able in Q.

($Conseq(Q)$ analogue of $Th(N)$ in Tarski's theorem — $N \models \varphi$ replaced by $Q \vdash \varphi$)

Pf (Sketch) Similar to Tarski
- First check that Q-rep'able sets are closed under complements and total recursive substitution
(if $\varphi(x)$ reps $R(x)$ then $\neg\varphi(\vec{x})$ reps R^c; if $\psi(\vec{x}, y)$ reps $f(\vec{x}) = y$ for f total recursive then $\exists k (\psi(\vec{x}, k) \wedge \varphi(k))$ reps $R(f(\vec{x}))$ — can you find the formal proofs?)
↳ implies, by a diagonal argument, there is no universal Q-rep'able set, i.e. no Q-rep'able $U \subseteq N^2$ s.t. for every Q-rep'able $B \subseteq N$ there is n s.t. $m \in B$ iff $(n, m) \in U$.
- if there were, could find φ s.t. $Q \vdash \varphi(\underline{n})$ and $Q \vdash \neg\varphi(\underline{n})$, contradicting consistency.

– leave details to you.

<u>Now:</u> Sps $\langle \mathrm{Conseq}(Q) \rangle$ is Q-rep'able, say by $\chi(x)$

We'll show there is a universal Q-rep'able set, contradiction

Define: $U(n,m)$ iff n codes a fmla $\varphi(x)$ and $Q \vdash \varphi(\underline{m})$

i.e. iff $\exists k\,(\mathrm{sub}(n,m) = k \wedge k \in \mathrm{Conseq}(Q))$

By Tarski-like arg, clear that U is universal. We claim it is Q-rep'able.

By hypothesis, $\mathrm{sub}(n,m) = k$ is Q-rep'able (since recursive), say by φ. Then

$$\tau(n,m) := \exists k\,(\varphi(n,m,k) \wedge \chi(k))$$

represents U. ✓ (Why, briefly: WTS $U(n,m) \Rightarrow Q \vdash \tau(\underline{n},\underline{m})$ and $\neg U(n,m) \Rightarrow Q \vdash \neg\tau(\underline{n},\underline{m})$

e.g. if $U(n,m)$ then if $k = \mathrm{sub}(n,m)$

$Q \vdash \varphi(\underline{n},\underline{m},\underline{k})$ and $Q \vdash \chi(\underline{k})$

hence $Q \vdash \varphi(\underline{n},\underline{m},\underline{k}) \wedge \chi(\underline{k})$ and hence

$Q \vdash \exists k\,(\varphi(\underline{n},\underline{m},k \wedge \chi(k))$

– assuming our deduction system can make these moves (it can)) ✓

Thus if Q reps all recursive relations / functions, follows from theorem that $\langle \mathrm{Conseq}(Q) \rangle$ is not recursive.

How rich need Q be, to rep all rec F's / R's?

We'll see there is a recursive (in fact, finite!) such Q:

Def'n Robinson's arithmetic Q consists of the following axioms (in language of arithmetic $L = \{<, S, +, \cdot, 0\}$)

(1) $\forall x\,(S(x) \neq 0)$
(2) $\forall x\,(x \neq 0 \Rightarrow \exists y\,(S(y) = x))$
(3) $\forall x \forall y\,(S(x) = S(y) \Rightarrow x = y)$
(4) "$<$ is a linear ordering"
i.e. $<$ is irreflexive, transitive, total
(5) $\forall x \forall y\,(x < S(y) \Rightarrow x = y \vee x < y)$
(6) $\forall x\,(0 \leq x)$
(7) $\forall x\,(x + 0 = x) \wedge \forall x \forall y\,(x + S(x) = S(x+y))$
(8) $\forall x\,(x \cdot 0 = 0) \wedge \forall x \forall y\,(x \cdot S(y) = x \cdot y + x)$

Theorem Every recursive function (and hence relation) is representable in Q_0

We'll need:

Lemma For every $m, n, p \in \mathbb{N}$ we have:

1. $Q_0 \vdash \forall x (x < \underline{n+1} \Rightarrow (x = \underline{0} \vee x = \underline{1} \vee \dots \vee x = \underline{n})$
2. if $m+n=p$, then $Q_0 \vdash \underline{m} + \underline{n} = \underline{p}$
3. if $m \cdot n = p$ then $Q_0 \vdash \underline{m} \cdot \underline{n} = \underline{p}$
4. if $m \neq n$ then $Q_0 \vdash \neg(\underline{m} = \underline{n})$

(we provide informal proofs from the axioms Q_0 and assume our deduction system is strong enough to turn these into formal proofs (it is))

(Think like this: can reason semantically as usual as though we're in *some* number system satisfying Q_0, but not necessarily the usual one – $\mathbb{N}$)

Pf: ① induction on n

for $n=0$, follows from Ax 5 (subbing 0 for y) + Ax 6. Sps for n. For $n+1$ we have (Ax 5) that $Q_0 \vdash \forall x (x < \underline{n+2} \Rightarrow x < \underline{n+1} \vee x = \underline{n+1})$ which by induction gives $Q_0 \vdash \forall x (x < \underline{n+2} \Rightarrow x = \underline{0} \vee x = \underline{1} \vee \dots \vee x = \underline{n+1})$

(2) induction on n using Ax 7
(3) induction on n using A 8.
(4) induction on m. If $m=0$ then $n \neq 0 \Rightarrow n = S(k)$ for some k (Ax 2) so ~~[illegible]~~ $Q_0 \vdash \underline{m} \neq \underline{n}$ by Ax 1. For $m+1$, if $n=0$ then again done by same arg, so assume ~~[illegible]~~ $n \neq 0$, so $n = k+1$ for some k, by Ax 2, so $m+1 \neq k+1$; so $m \neq k$ by Ax 3

really $S(k)$

By induction $Q_0 \vdash \underline{m} \neq \underline{k}$

so $Q_0 \vdash \underline{m+1} \neq \underline{k+1}$ by Ax 3.

$= \underline{n}$ ✓

Pf of thm: We show every min-recursive function is representable
By HW: min-recursive $\Leftrightarrow$ recursive
so this suffices

Recall: a (partial) $f : \mathbb{N}^k \to \mathbb{N}$ is min-rec iff built up from basic functions:

$x \mapsto 0$

$+$

$\cdot$

$(x_1, \dots, x_k) \to x_i$ for each $i \leq k$

$\chi_=$ (char. function of $=$)

using composition and minimalization

Recall: to check $f(x) = y$ is rep'able by φ, WTS $Q_0 \vdash \forall x, y, z\ (\varphi(x,y) \wedge \varphi(x,z) \Rightarrow y = z)$ and $Q_0 \vdash \varphi(\underline{n}, \underline{m})$ whenever $f(n) = m$.

Basic functions above are Q_0-rep'able, as follows:

(i) $x \to 0$ is rep'able by $\varphi(x,y) := y = 0$. Why: $Q_0 \vdash \forall x, y, z\ (y = 0 \wedge z = 0 \Rightarrow y = z)$ follows from axioms for $=$ in our deduction system — don't need any axioms from Q_0

Also $Q_0 \vdash \varphi(\underline{x}, \underline{0})$ for all $x \in \mathbb{N}$, since this is just statement $0 = 0$ (since $\underline{0}$ is 0), which also follows from $=$ axioms.

(ii) $+$ is rep'able by $\varphi(x_1, x_2, y) := x_1 + x_2 = y$. Unique output property holds by transitivity of $=$. And if $n + m = k$ then $Q_0 \vdash \underline{n} + \underline{m} = \underline{k}$ by lemma (2) above

(iii) $\cdot$ rep'able by $\varphi(x_1, x_2, y) := x_1 \cdot x_2 = y$. Proof similar, now use lemma (3) (iv.) $(x_1, .., x_k) \to x_i$ rep'ble by $\varphi(x_1, .., x_k, y) := y = x_i$. Just need properties of $=$ to prove φ works

(v) $\chi_=$ rep'ble by $\varphi(x_1, x_2, y) := (x_1 = x_2 \wedge y = \underline{1}) \vee (x_1 \neq x_2 \wedge y = \underline{0})$.

If $\varphi(x_1, x_2, y)$ and $\varphi(x_1, x_2, z)$ then can prove $y = z$ using Ax 1 (i.e. $S(0) \neq 0$ i.e. $\underline{1} \neq \underline{0}$)

And if $\chi_=(x_1, x_2) = e$, then $Q_0 \vdash \varphi(\underline{x_1}, \underline{x_2}, \underline{e})$ by Lemma (4). (can prove $\underline{x_1} = \underline{x_2}$ if $x_1 = x_2$ for free... just need to worry about $x_1 \neq x_2$ case)

Now we must show rep'ble functions closed under composition + minimalization.

<u>Composition</u>: Sps g and $h_1, .., h_k$ are rep'ble, say by ψ, and $\varphi_1, .., \varphi_k$ and $f = g(h_1(x_1, .., x_n), .., h_k(x_1, .., x_n))$

Then f is rep'ble by $\tau(\vec{x}, y) :=$

$$\exists z_1 .. \exists z_k (\varphi_1(\vec{x}, z_1) \wedge .. \wedge \varphi_k(\vec{x}, z_k) \wedge \psi(z_1, .., z_k, y)$$

Unique output property follows from u.o. prop. of φ_i and ψ.

And if $f(a_1, \ldots, a_n) = m$ we know $Q_0 \vdash \varphi_i(\underline{\vec{a}}) = \underline{m_i}$ and $Q_0 \vdash \psi(\underline{m_1}, \ldots, \underline{m_k}, \underline{m})$ so the m_i's witness the $\exists z_i$'s and $Q_0 \vdash \tau(\underline{a_1}, \ldots, \underline{a_n}, \underline{m})$ ✓

<u>minimalization</u>: Assume $g: \mathbb{N}^{n+1} \to \mathbb{N}$ is rep'able by ψ and $f(x_1, \ldots x_n) = \mu y[g(x_1, \ldots, x_n, y) = 0]$.

Then f is rep'able by $\varphi(\vec{x}, y) :=$
$$\psi(\vec{x}, y, 0) \wedge \underbrace{\forall w(w<y}_{\text{"}\forall w<y\text{"}} \Rightarrow \exists z(z \neq 0 \wedge \psi(\vec{x}, w, z))$$

Then if $\varphi(\vec{x}, y)$ and $\varphi(\vec{x}, z)$ we have $y \geq z$ and $z \geq y$ so by Ax 4 ("$<$ is linear order") we have (and $Q_0 \vdash$) $y = z$.

If $f(a_1, \ldots, a_n) = m$ then $g(a_1, \ldots, a_n, m) = 0$ and $g(\vec{a}, i) = l_i \neq 0$ $\forall i < m$

<u>WTS</u>: $Q_0 \vdash \varphi(\underline{a_1}, \ldots, \underline{a_n}, \underline{m})$ i.e.

$Q_0 \vdash \psi(\underline{a_1}, \ldots, \underline{a_n}, \underline{m}, 0)$ (which is true since ψ reps g), and:

$Q_0 \vdash \forall w(w < \underline{m} \Rightarrow \exists z(z \neq 0 \wedge \psi(\underline{\vec{a}}, w, z))$