— though we'll be interested primarily in variations of language of arithmetic, we develop coding wrt a general countable language (that we fix)

$$L = \{R_i\} \cup \{f_j\} \cup \{c_k\}$$

<u>Coding symbols</u>: we write $\langle \ldots \rangle$ to mean "code for ..."

$\langle \ldots \rangle$ is the <u>Gödel number</u> of ...

Recall our language also includes the logical symbols — for convenience (given our deductive system) let's assume there are $\neg$, $\Rightarrow$, $\forall$, $($, $)$, "comma", $=$ and the variables $x_0, x_1, \ldots$

We code non-variable logical symbols as binary codes w/ leading 0:

$\langle \neg \rangle = \langle 0, 0 \rangle$ — abusing $\langle \rangle$'s here by $\langle a, b \rangle$ we mean in old sense: $2^{a+1} 3^{b+1}$

$\langle \Rightarrow \rangle = \langle 0, 1 \rangle$

$\langle \forall \rangle = \langle 0, 2 \rangle$

$\vdots$

$\langle = \rangle = \langle 0, 6 \rangle$

Code ~~[illegible]~~ x_n, c_k, R_i, f_j as follows (say):

$$\langle R_n \rangle = \langle 0, 7+4n \rangle$$
$$\langle f_n \rangle = \langle 0, 7+4n+1 \rangle$$
$$\langle c_n \rangle = \langle 0, 7+4n+2 \rangle$$
$$\langle x_n \rangle = \langle 0, 7+4n+3 \rangle$$

Coding terms: Inductively, if $t = f(t_1, \ldots, t_m)$ and t_j are already encoded terms, define
$$\langle t \rangle = \langle 1, \langle f \rangle, \langle t_1 \rangle, \ldots, \langle t_m \rangle \rangle$$

Formulas Inductively define:
$$\langle R(t_1, \ldots, t_m) \rangle = \langle 2, \langle R \rangle, \langle t_1 \rangle, \ldots, \langle t_m \rangle \rangle$$
$$\langle t = s \rangle = \langle 2, \langle = \rangle, \langle t \rangle, \langle s \rangle \rangle$$
and then (e.g.)
$$\langle \neg \varphi \rangle = \langle 2, \langle \neg \rangle, \langle \varphi \rangle \rangle$$
etc. for $\langle \varphi \Rightarrow \chi \rangle$, $\langle \forall x_i \varphi \rangle$

Sequences of formulas: If $\varphi_0, \ldots, \varphi_n$ is a sequence of fmlas define
$$\langle \varphi_0, \ldots, \varphi_n \rangle = \langle \langle \varphi_0 \rangle, \ldots, \langle \varphi_n \rangle \rangle$$

Now we define the relations:

Var(n) iff n is Gödel # of a variable
Term(n) " " " term
Form(n) " " " formula
Sent(n) " " sentence
LogicAx(n) " " logical axiom
MP(k, l, m) iff Form(k) ∧ Form(l) ∧ Form(m) ∧ (formula coded by m follows from formulas coded by k, l by modus ponens)
Proof(n) iff n is Gödel # for a proof (in our deductive system)

- These sets will depend on the particular language we fix
- But for a fixed L they are "clearly" recursive (in fact prim. recursive: need to do iterative checking but can bound # of steps in e.g. n)
- Hence they are definable in $\mathbb{N}$.

For example, Proof(n) holds iff

$$Seq(n) \wedge \forall i < length(n) \left[Form((n)_i) \wedge \left[LogicAx((n)_i) \vee \exists j,k < i (MP((n)_j,(n)_k,(n)_i)) \right] \right]$$

which is p.r. (... assuming Form, LogicAx, MP are ... even for Term(n) need a nontrivial primitive recursion)

<u>Def'n</u> Spse Φ is a collection of fmlas. Say Φ is <u>recursive</u> if

$$\langle \Phi \rangle := \{ \langle \varphi \rangle : \varphi \in \Phi \} \text{ is recursive.}$$

as a subset of $\mathbb{N}$.

<u>Def'n</u> Given a set of fmlas Φ, <u>define</u> $Proof_\Phi(n)$ iff n is a code for a proof from formulas in Φ (i.e. $Proof(n) \wedge \forall i < length \left[LogicAx((n)_i) \vee (n)_i \in \langle \Phi \rangle \right]$)

<u>Observe</u>: $Proof_\Phi$ is recursive if Φ is recursive

<u>Def'n</u> Sps Σ is a collection of fmlas. We define
$$\mathrm{Conseq}(\Sigma) = \{\varphi : \Sigma \vdash \varphi\}$$

<u>Def'n</u> We say that a set of formulas Σ is <u>complete</u> if for every φ, either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$
(i.e. either $\varphi \in \mathrm{Conseq}(\Sigma)$ or $\neg\varphi \in \mathrm{Conseq}(\Sigma)$)

<u>Prop'n</u> Sps Σ is a collection of fmlas. If Σ is recursive then $\mathrm{Conseq}(\Sigma)$ is r.e. Moreover, if Σ is complete then $\mathrm{Conseq}(\Sigma)$ is recursive.

<u>Pf</u>: Observe
$$n \in \ulcorner \mathrm{Conseq}(\Sigma) \urcorner \text{ iff } \exists m \, (\mathrm{Proof}_\Sigma(m) \wedge (m)_{\mathrm{length}(m)-1} = n)$$

$\Rightarrow \ulcorner \mathrm{Conseq}(\Sigma) \urcorner$ is r.e. since Proof_Σ is recursive. (as Σ is)

Now sps further Σ is complete. If Σ proves both φ and $\neg\varphi$ for some φ we know $\mathrm{Conseq}(\Sigma) =$ all fmlas, which is recursive

So sps Σ is consistent. Then by completeness we have $\varphi \in \mathrm{Conseq}(\Sigma)$ iff Σ cannot prove $\neg\varphi$, i.e.

$$n \in \langle \mathrm{Conseq}(\Sigma) \rangle \text{ iff } \neg \exists m \left(\mathrm{Proof}_\Sigma(m) \wedge (m)_{\mathrm{length}(m)-1} = \mathrm{neg}(n) \right)$$

Where $\mathrm{neg}(n)$ denotes code of negation of fmla coded by n (is p.r.)

$\Rightarrow \langle \mathrm{Conseq}(\Sigma) \rangle$ is co-r.e. in this case too.

$\Rightarrow \langle \mathrm{Conseq}(\Sigma) \rangle$ is recursive ✓

We can now prove a famous theorem of Tarski that says that the (set of codes for the) full theory of $\mathcal{N}$ is a complicated set.

$\mathcal{N} = \langle N, 0, <, S, +, \cdot \rangle$

<u>Def'n</u> $\mathrm{Th}(\mathcal{N}) := \{\varphi : \mathcal{N} \models \varphi\}$

the "theory of $\mathcal{N}$"

(Here we assume the language we fixed is lang of arithmetic ... or maybe some extension of it.)

<u>Theorem</u> (Tarski) $\langle Th(\mathcal{N}) \rangle$ is not arithmetical.

We'll need the following:

<u>Lemma</u>: There is no universal arithmetical set, i.e., there is no arithmetical set (i.e. $\in \bigcup_n \Delta^0_n$) $U \subseteq \mathbb{N}^2$ s.t. for every arithmetical $B \subseteq \mathbb{N}$ there is n s.t.

$$m \in B \Leftrightarrow (n,m) \in U.$$

used diag.

<u>Pf</u>: Since each Δ^0_n is closed under complements, so is $\bigcup \Delta^0_n$. If we had such a U, then

Then $W = \{n : (n,n) \in U\}$ is arithmetical, Hence so is W^c.

Hence $\exists k$ s.t. $m \in W^c$ iff $(k,m) \in U$

So then $k \in W^c$ iff $(k,k) \in U$ iff $k \in W$, contradiction ✓

Pf of Tarski: Sps $\langle Th(\mathcal{N})\rangle$ is arithmetical. Define $\mathcal{U}$ by:

$(n,m) \in \mathcal{U}$ iff

"n codes a formula of the form $\varphi(x)$" and "$\mathcal{N} \models \varphi(\underline{m})$"

"clearly" recursive $\qquad \varphi(\underline{m})$

i.e. the relation

$Sub(n,m) = k$ iff n codes a formula $\varphi(x)$ and k codes $\varphi(\underline{m})$ is recursive ("clearly")

recall: $\underline{m}$ denotes term: $S(S(\cdots S(0)\cdots))$

So can write def'n of $\mathcal{U}$ as

$\exists k(Sub(n,m) = k \wedge k \in \langle Th(\mathcal{N})\rangle)$

which makes clear $\mathcal{U}$ is arithmetical since $\langle Th(\mathcal{N})\rangle$ is.

We claim $\mathcal{U}$ is universal, which gives a contradiction:

If $B \subseteq \mathbb{N}$ is arithmetical then (as we showed previously!) there is a fmla $\varphi(x)$ s.t.

$m \in B$ iff $\mathcal{N} \models \varphi(\underline{m})$

But then $(\langle\varphi\rangle, m) \in U$ iff $\varphi(\underline{m}) \in$ Th($\mathcal{N}$) iff $\mathcal{N} \models \varphi(\underline{m})$ iff $m \in B$

$\Rightarrow$ U is universal, contradiction ✓

From Tarski's theorem we can deduce a weak form of Gödel's 1st Incompleteness Theorem.

<u>Theorem</u> (1st incomp. thm weak form)
Sps Σ is a collection of fmlas (in lang. of arithmetic)

cannot axiomatize Th($\mathcal{N}$) recursively

If $\mathcal{N} \models \Sigma$ (so in particular Σ is consistent) and Σ is recursive, then Σ is not complete.

<u>Pf.</u> We showed $\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi$ so that we always have Conseq(Σ) $\subseteq$ Th($\mathcal{N}$).

Th($\mathcal{N}$) is complete in stronger sense: either $\varphi \in$ Th($\mathcal{N}$) or $\neg\varphi \in$ Th($\mathcal{N}$) for every φ (since exactly one true in $\mathcal{N}$)

If Σ complete then would have Conseq(Σ) = Th($\mathcal{N}$). But then Th($\mathcal{N}$) is r.e. by above, contradicting Tarski ✓

Hence there are φ s.t. $\Sigma \nvdash \varphi$ and $\Sigma \nvdash \neg\varphi$. Such φ are called <u>independent</u> from Σ

<u>Toward Gödel's Theorems</u>

- Gödel's original 1st incomp. thm is stronger than the one above.
- Not based on notion of <u>definability</u> which depends on semantic concept $\mathcal{N} \models \varphi(m)$
- But rather "representability" which is syntactic: doesn't make reference to a specific structure in our lang. such as $\mathcal{N}$.

Fix a lang. L containing constant symbol 0 and unary function symbol S — so the terms

$$\underline{m} := \underbrace{S(S(\cdots(S(0))\cdots))}_{m \text{ times}}$$

belong to L.

Let Q be any fixed collection of ~~[illegible]~~ sentences.

- A subset $R \subseteq \mathbb{N}^n$ is <u>representable in Q</u> if there is a formula φ s.t. for each $x_1, \dots, x_n \in \mathbb{N}$

$$R(x_1, \dots, x_n) \Rightarrow Q \vdash \varphi(\underline{x_1}, \dots, \underline{x_n})$$

$$\neg R(x_1, \dots, x_n) \Rightarrow Q \vdash \neg \varphi(\underline{x_1}, \dots, \underline{x_n})$$

- A (partial) function $f: \mathbb{N}^n \to \mathbb{N}$ is <u>representable in Q</u> if there is a $\varphi(x_1, \dots, x_n, x)$ s.t.

$$Q \vdash \forall x_1, \dots, x_n, y, z \left((\varphi(x_1, \dots, x_n, y) \wedge \varphi(x_1, \dots, x_n, z)) \Rightarrow y = z \right)$$

and $f(x_1, \dots, x_n) = y \Rightarrow Q \vdash \varphi(\underline{x_1}, \dots, \underline{x_n}, \underline{y})$

<u>Observe:</u> if Q is collection of formulas true in $\mathcal{N}$ (i.e. $\mathcal{N} \models Q$) (think: Q is attempt to axiomatize $Th(\mathcal{N})$) then R being Q-representable $\Rightarrow$ R is definable, since

$$Q \vdash \varphi(\vec{x}) \Rightarrow \mathcal{N} \models \varphi(\vec{x})$$

$$Q \vdash \neg\varphi(\vec{x}) \Rightarrow \mathcal{N} \not\models \varphi(\vec{x})$$

Theorem Sps $L \supseteq \{0, S\}$ and Q a set of sentences (our axioms). Sps every recursive relation and function is Q-representable. Then $\langle Conseq(Q) \rangle$ is not Q-representable.

Pf. Similar to Tarski. First check the Q-representable sets are closed under complement and use this to prove there is no universal Q-representable set (exercise)

i.e. no representable $U \subseteq \mathbb{N}^2$ s.t. for every rep'ble $B \subseteq \mathbb{N}$ $\exists n$ s.t. $m \in B$ iff $(n,m) \in U$

- Sps $\langle Conseq(Q) \rangle$ is Q-representable say by a fmla ψ.
- We'll show there is a universal Q-representable U, contradiction

Define $U(n,m)$ iff n codes a fmla $\varphi(x)$ and $Q \vdash \varphi(\underline{m})$

i.e. iff $\exists k \, (Sub(n,m) = k \wedge k \in Conseq(Q))$

By a similar arg. to Tarski, clear that U is universal.

We claim U is representable