Deductive Systems and the Completeness Theorem

A deductive formal system $\underset{\sim}{D}$ for a language L consists of
- a collection of formulas in the language L ("axioms")
- a finite set of rules of inference that is, a finite set of relations $D_i(\varphi_1, \ldots, \varphi_{n_i}, \chi)$ on the formulas in L.

$D_i(\varphi_1, \ldots, \varphi_{n_i}, \chi)$ intuitively means χ can be deduced from $\varphi_1, \ldots, \varphi_{n_i}$ by applying the deduction rule D_i.

We say χ is a direct consequence of $\varphi_1, \ldots, \varphi_{n_i}$ if $\vdash D(\varphi_1, \ldots, \varphi_{n_i}, \chi)$ for some i.

A formal proof is a sequence of formulas s.t. each formula is either an axiom or a direct consequence of some formulas preceding it.

φ is a formal theorem if there is a formal proof $\varphi_1, \ldots, \varphi_m$

with $\varphi_m = \varphi$. We write $\vdash_{\mathcal{D}} \varphi$ (or just $\vdash \varphi$) if there is such a proof.

If Σ is a collection of formulas in $\mathcal{L}$ then a formal proof from Σ is the same as before, except now we're allowed to use formulas from Σ as well as axioms in our proof. We write $\Sigma \vdash \varphi$.

Clearly we have the following facts: ① if $\Sigma \subseteq \Sigma'$ and $\Sigma \vdash \varphi$ then $\Sigma' \vdash \varphi$

② If $\Sigma \vdash \varphi$ then for some finite $\Sigma_0 \subseteq \Sigma$ we have $\Sigma_0 \vdash \varphi$ (proofs are finite length!)

③ If $\Sigma' \vdash \varphi$ and for all $\chi \in \Sigma'$ we have $\Sigma \vdash \chi$, then $\Sigma \vdash \varphi$.

A formal deductive system for first-order logic

We describe a deductive system w/ just a single deduction rule (modus ponens)
- Works for any language
- Fix a language L.
- if φ is an L-formula, a generalization of φ is any fmla of the form $\forall x_1 \forall x_2 \cdots \forall x_k \varphi$.

We take as axioms all generalizations of all formulas of the following forms:

(1) Propositional axioms:
(a.) $\varphi \Rightarrow (\chi \Rightarrow \varphi)$
(b.) $(\varphi \Rightarrow (\chi \Rightarrow \eta)) \Rightarrow ((\varphi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \eta))$
(c.) $(\neg\varphi \Rightarrow \neg\chi) \Rightarrow ((\neg\varphi \Rightarrow \chi) \Rightarrow \varphi)$

(2) Quantifier axioms:
(a.) $\forall x(\varphi(x) \Rightarrow \chi(x)) \Rightarrow (\forall x \varphi(x) \Rightarrow \forall x \chi(x))$
(b) $\varphi \Rightarrow \forall x \varphi$, if x not free in φ
(c.) $\forall x \varphi(x) \Rightarrow \varphi(x|t)$

↑ formula in which all free instances of x replaced w/ t

if t is a term substitutable for x, i.e. no free occurrence of x is in scope of $\forall z$ for some z appearing in t

(e.g. if t is $x+y$ then $\forall x(x=x)$
$\Rightarrow (x+y=x+y)$ is an axiom, whereas
$\forall x \exists y\,(x=y+1) \Rightarrow \exists y\,(x+y=y+1)$
is not — t not subst'able for x in latter)

Equality axioms:

(a) $x=x$;
$x=y \Rightarrow y=x$;
$x=y \wedge y=z \Rightarrow x=z$

(b) For every n-ary rel'n symbol R:
$(x_1=y_1 \wedge \dots \wedge x_n=y_n) \Rightarrow$
$[R(x_1,\dots,x_n) \Rightarrow R(y_1,\dots,y_n)]$

(c) For every n-ary function symbol f:
$(x_1=y_1 \wedge \dots \wedge x_n=y_n) \Rightarrow$
$[f(x_1,\dots,x_n) = f(y_1,\dots,y_n)]$.

(All intuitive axioms.
Why enough to get by? Have to try to see.
Missing some you might think, e.g.
$\varphi \Rightarrow \varphi$ or $\varphi \Rightarrow \neg\neg\varphi$
But these can be proved.)

(Easy to see: all axioms are tautologies
i.e. if t is an axiom and $\mathcal{A}$ is an L-structure
then $\mathcal{A} \models t$.)

We use only one deduction rule:
$D_1(\varphi, \varphi \Rightarrow \chi, \chi)$

(i.e. From φ, $\varphi \Rightarrow \chi$ can conclude χ)

Write MP instead of D_1.

↳ using a single deduction rule is nice, but makes some proofs awkward since we don't have direct deductions for formulas written using $\wedge$'s, $\vee$'s, etc.

Note: axioms and deduction rule are syntactic things.

Ex: Spse $\Sigma \vdash \chi$ and $\Sigma \vdash \neg\chi$ for some χ.
Then Σ proves every formula.
$\vdash \neg\chi$ a fmla φ.

Pf: (1) (1.a) $\chi \Rightarrow (\neg\varphi \Rightarrow \chi)$
(2) (1.a) $\neg\chi \Rightarrow (\neg\varphi \Rightarrow \neg\chi)$
(3) $\Sigma : \chi$
(4) $\Sigma : \neg\chi$
(5) MP(1,3): $\neg\varphi \Rightarrow \chi$
(6) MP(2,4) $\neg\varphi \Rightarrow \neg\chi$
(7) (1.c) $(\neg\varphi \Rightarrow \neg\chi) \Rightarrow ((\neg\varphi \Rightarrow \chi) \Rightarrow \varphi)$

(5) MP(5,7): $(\neg\varphi \Rightarrow \neg\psi) \Rightarrow \psi$
(9) MP(6,8): ψ

More examples: (a.) $\vdash \varphi \Rightarrow \varphi$
(b.) $\vdash \neg\neg\varphi \Rightarrow \varphi$

Pf: you try.

Lemma (Deduction Lemma): If $\Sigma \cup \{\varphi\} \vdash \psi$ then $\Sigma \vdash \varphi \Rightarrow \psi$

Pf: Let $\varphi_1, \ldots, \varphi_n$ be a deduction from $\Sigma \cup \{\varphi\}$.

We check, $\forall i \leq n$, we have $\Sigma \vdash \varphi \Rightarrow \varphi_i$ (so in particular if this is a deduction of $\varphi_n = \psi$ we are done)

If not, find least i s.t. not. Three possibilities:

(1) φ_i is an axiom or in Σ. Then by (1.a) we get $\varphi_i \Rightarrow (\varphi \Rightarrow \varphi_i)$ is an axiom and by MP get $\varphi \Rightarrow \varphi_i$. (i.e. $\Sigma \vdash \varphi \Rightarrow \varphi_i$)

(2) φ_i is φ. Then by example (a.) above we get $\vdash \varphi \Rightarrow \varphi_i$. So $\Sigma \vdash \varphi \Rightarrow \varphi_i$

(3) φ_i is deduced by applying MP to formulas φ_k, $\varphi_k \Rightarrow \varphi_i$ appearing previously in deduction

By minimality of i we get
$\Sigma \vdash \varphi \Rightarrow \varphi_k$ and $\Sigma \vdash \varphi \Rightarrow (\varphi_k \Rightarrow \varphi_i)$

By (1b.) we get
$(\varphi \Rightarrow (\varphi_k \Rightarrow \varphi_i)) \Rightarrow ((\varphi \Rightarrow \varphi_k) \Rightarrow (\varphi \Rightarrow \varphi_i))$

By MP: $(\varphi \Rightarrow \varphi_k) \Rightarrow (\varphi \Rightarrow \varphi_i)$

By MP: $\varphi \Rightarrow \varphi_i$, contradiction ✓

<u>Theorem</u> (Gödel's completeness theorem, first form) For any Σ and φ we have

$$\Sigma \models \varphi \text{ iff } \Sigma \vdash \varphi$$

(Recall: $\Sigma \models \varphi$ means if $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \varphi$ for every L-structure $\mathcal{A}$.

Also, if φ has free variables $x_1, \dots x_n$ then $\mathcal{A} \models \varphi$ means $\mathcal{A} \models \forall x_1 \dots \forall x_n \varphi$)

($\Leftarrow$) is straightforward direction — "soundness" — if Σ proves φ then φ is true in every model of φ

($\Rightarrow$) is subtler — "completeness" — if φ true in every model, can actually be proved.

Pf of soundness: Sps $\Sigma \vdash \varphi$ and $\mathcal{A} \models \Sigma$.
WTS $\mathcal{A} \models \varphi$.

Key point: (also clear): in general, if $\mathcal{A} \models \tau(\bar{a})$ for any tuple $\bar{a}$ and also $\mathcal{A} \models \tau(\bar{a}) \Rightarrow \sigma(\bar{a})$ $\forall \bar{a}$ then $\mathcal{A} \models \sigma(\bar{a})$ $\forall \bar{a}$.

Sps $\varphi_1, \ldots, \varphi_n$ is a deduction of $\varphi = \varphi_n$ from Σ. We prove $\mathcal{A} \models \varphi_i$ $\forall i \leq n$.
If not, let i be least s.t. $\mathcal{A} \not\models \varphi_i$.
- if φ_i is an axiom then it is a tautology hence $\mathcal{A} \models \varphi_i$, contradiction
- if $\varphi_i \in \Sigma$ then $\mathcal{A} \models \varphi_i$ by hypothesis, contradiction
- if φ_i is deduced by MP from φ_k, $\varphi_k \Rightarrow \varphi_i$ then by minimality of i we have $\mathcal{A} \models \varphi_k$, $\mathcal{A} \models \varphi_k \Rightarrow \varphi_i$
hence $\mathcal{A} \models \varphi_i$ by observation above, a contradiction ✓

Before considering $\Rightarrow$ let's see some consequences.

Corollary (Compactness theorem) If $\Sigma \models \varphi$ then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \models \varphi$.

Pf: By completeness we know $\Sigma \vdash \varphi$. By finiteness of proofs there is $\Sigma_0 \subseteq \Sigma$ finite s.t. $\Sigma_0 \vdash \varphi$. Then $\Sigma_0 \models \varphi$ by soundness. ✓

We can reformulate the completeness theorem as follows.

Defn Σ is consistent if there is no sentence φ s.t. $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$.

By example, Σ is inconsistent iff Σ proves every formula.

Observe: if Σ inconsistent, Σ has no models (can't have $\Delta \models \varphi$ and $\Delta \models \neg\varphi$, by defn of $\models$)

Converse also true!

Theorem (Completeness theorem, 2nd form) Σ is consistent iff Σ has a model.

Prop'n Two forms of the completeness theorem are equivalent

(I mean the hard directions, i.e. $\Sigma \vdash \varphi \Rightarrow \Sigma \vdash \varphi$ (if Σ consistent $\Rightarrow \Sigma$ has a model)

Pf: Sps 1st form, and suppose $\Sigma \cup \{\neg\varphi\}$ consistent. If it had no model then for any ψ, $\Sigma \vdash \psi$ and $\Sigma \vdash \neg\psi$, trivially. Hence by 1st form of completeness we have $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$, contradicting consistency.

Sps 2nd form and suppose $\Sigma \vdash \varphi$ then $\Sigma \cup \{\neg\varphi\}$ has no model. Hence $\Sigma \cup \{\neg\varphi\}$ inconsistent by 2nd form. Hence $\Sigma \cup \{\neg\varphi\} \vdash \chi$ and $\vdash \neg\chi$ for any χ.

But then by deduction lemma, $\Sigma \vdash \neg\varphi \Rightarrow \chi$ and $\Sigma \vdash \neg\varphi \Rightarrow \neg\chi$. But then by axiom (1c) and MP (applied twice) we get $\Sigma \vdash \varphi$ ✓

Reformulation of completeness in this way led to a more elegant proof than Gödel's original

How to show: if Σ consistent then Σ has a model? Build a "Henkin model"! (Won't do...)

Arithmetization of syntax:

Previous section: there is as-good-as-possible correspondence between truth-in-models (semantic notion) and formulas/proofs (syntactic notion)

Since fmlas/pfs are syntactic, would like to make sense of sets of fmlas/pfs being recursive, r.e., etc.

is a key step toward proving Incompleteness theorems.

Need to code fmlas/pfs as natural numbers! (effectively...)