Ma 116a Homework #5

Due Friday, March 8th at 1:00pm

1) Suppose that X and Y are sets, and $\mathbb{P} = \{p : p \text{ is a finite partial function from } X \text{ to } Y\}$ ordered by $p \leq q$ if $p \supseteq q$. Show that if Y is countable, then $\mathbb{P}$ has the ccc.

   Hint: Fill in the details of the following argument. Suppose $A \subseteq \mathbb{P}$ is uncountable. We show A is not an antichain. By the Δ-system lemma we can find $D \subseteq A$ an uncountable Δ-system with root r. Argue that distinct conditions $p, q \in D$ are compatible iff $\mathrm{dom}(p \setminus r) \cap \mathrm{dom}(q \setminus r) = \emptyset$. For $p \in D$, let $d_p = \mathrm{dom}(p \setminus r)$. Let $Dom = \{d_p : p \in D\}$. Argue that Dom must be uncountable, and then argue there must be $d_p, d_q \in Dom$ with $d_p \cap d_q = \emptyset$.

2) For functions $f, g \in {}^{\omega}\omega$, define $f <^* g$ if there is $n \in \omega$ such that for all $m > n$ we have $g(m) > f(m)$.

   i. Show that if $\mathscr{F} \subseteq {}^{\omega}\omega$ is countable, then there is $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$.

   For the remaining parts of the problem, suppose that MA holds and $\mathscr{F} \subseteq {}^{\omega}\omega$ has size $\kappa < 2^{\omega}$. Our goal is to show there is $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$.

   ii. Let $\mathbb{P}$ consist of all conditions $\langle p, F\rangle$ where p is a finite partial function from ω to ω and F is a finite subset of $\mathscr{F}$. Order $\mathbb{P}$ by the rule $\langle p, F\rangle \leq \langle q, G\rangle$ iff $p \supseteq q$, $F \supseteq G$, and $\forall f \in G$, if $n \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ then $p(n) > f(n)$. Show that $\mathbb{P}$ has the ccc.

   Hint: Argue that two conditions of the form $\langle p, F\rangle$ and $\langle p, G\rangle$ are always compatible, and use this to prove the claim.

   iii. Show that for every $f \in \mathscr{F}$, the set $D_f \subseteq \mathbb{P}$ consisting of all conditions $\langle p, F\rangle$ with $f \in F$, is dense in $\mathbb{P}$. Show that D_n is also dense, where D_n is the set of conditions $\langle p, F\rangle$ with $n \in \mathrm{dom}(p)$.

   iv. By MA there is a filter G intersecting D_f and D_n for every $f \in \mathscr{F}$ and $n \in \omega$. Show that

   $$g = \bigcup\{p : \langle p, F\rangle \in G \text{ for some finite } F \subseteq \mathscr{F}\}$$

   is as desired.