Ma 116a Homework #4

Due Thursday, February 29th at 1:00pm

1) Suppose κ is a regular uncountable cardinal. Show that if $C \subseteq \kappa$ is club and $S \subseteq \kappa$ is stationary then $C \cap S$ is stationary.

2) Suppose T is a tree. Recall that a *subtree* of T is a subset $S \subseteq T$ such that for all $s \in S$ and $t \in T$, if $t \leq s$ then $t \in S$. A *branch* $B \subseteq T$ is a maximal linearly ordered subset of T.

   Given a subtree $S \subseteq T$, its *outer boundary* $\partial(S)$ is $\{t \in T \setminus S : \exists s \in S(s < t) \wedge \nexists t' \in T(s < t' < t)\}$.

   i. Prove the following extension of König's lemma: if T is an ω-tree such that for every nonempty finite subtree $S \subseteq T$ we have $|\partial(S)| \geq 2$, then T has at least two infinite branches.

      You may assume T is rooted, i.e. $|\mathrm{Lev}_0(T)| = 1$. This is not necessary, but simplifies the argument.

   ii. Show by example that the hypothesis "for every finite subtree $S \subseteq T$ we have $|\partial(S)| \geq 2$" cannot be replaced by "for every $n \in \omega$ we have $|\mathrm{Lev}_n(T)| \geq 2$" in the previous part.

3) Show that any Aronszajn tree T that is a subtree of $\{s \in {}^{<\omega_1}\omega : s \text{ is one-to-one}\}$ cannot be Suslin. (In particular, the Aronszajn tree we constructed in class is not Suslin.)

   Hint: You have to find an uncountable antichain in T. Show that for each $n \in \omega$, the collection $A_n = \{s \in T : \exists \alpha \in \omega_1(\mathrm{dom}(s) = \alpha + 1 \wedge s(\alpha) = n)\}$ is an antichain. Why is this sufficient?

4) In this problem we give another proof (due to Banach) of the existence of a stationary and co-stationary $S \subseteq \omega_1$.

   For $x, y \in \mathbb{R}$, we write $[\{x, y\}]$ for the closed interval with endpoints x, y. (If $x = y$, then $[\{x, y\}] = [x, x] = \{x\}$, which we consider a closed interval.)

   Fix an injection $f : \omega_1 \to \mathbb{R}$. Define a relation $\sim$ on $\mathbb{R}$ by the rule $x \sim y$ if $f^{-1}[\{x, y\}]$ is non-stationary.

   i. Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.

   ii. Prove that in fact $\sim$ is a *convex* equivalence relation, that is, if $x < y < z$ and $x \sim z$, then $x \sim y \sim z$. Conclude that for every $x \in \mathbb{R}$, the equivalence class $[x]$ is an interval.

   iii. Show that there are at least three $\sim$-equivalence classes.

      Hint: If not, then either $\mathbb{R}$ is an equivalence class, or $\mathbb{R}$ can be written as a disjoint union of equivalence classes $I \cup J$, where I is an initial segment of $\mathbb{R}$ and J is the corresponding final segment. Show that in either case ω_1 can be written as a countable union of non-stationary sets.

   iv. By (iii.), we can find real numbers $x < y < z$ such that $x \not\sim y \not\sim z$. Then $f^{-1}[[x, y]]$ and $f^{-1}[[y, z]]$ are stationary. Argue $f^{-1}[(y, z]]$ is also stationary. Conclude $S = f^{-1}[[x, y]]$ is stationary and co-stationary.