Ma 116a Homework #3

Due Thursday, February 15th at 1:00pm

1) Suppose κ is a cardinal such that $\mathrm{cf}(\kappa) > \omega$ and $A \subseteq \kappa$.

   - An ordinal $\alpha \in \kappa$ is a *limit point* of A if $\alpha = \sup B$ for some subset $B \subseteq A$.
   - A is *closed* if whenever α is a limit point of A we have $\alpha \in A$.
   - A is *unbounded* if for every $\alpha \in \kappa$ there is $\beta \in A$ such that $\beta \geq \alpha$.
   - A is a *club* if A is both closed and unbounded.

   Argue briefly that $A_0 = \{\alpha \in \kappa : \exists\delta(\alpha = \omega^\delta)\}$ is a club and $A_1 = \{\alpha \in \kappa : \exists\delta(\alpha = \delta + 1)\}$ is not a club. Show that if A and B are clubs in κ then $A \cap B$ is also a club.

   Some culture: By an only superficially more complicated argument one can show that the intersection of fewer than $\mathrm{cf}(\kappa)$-many clubs is a club. That is, for every cardinal $\lambda < \mathrm{cf}(\kappa)$, if $\{A_\delta : \delta < \lambda\}$ is a collection of club subsets $A_\delta \subseteq \kappa$ indexed by λ, then $A = \bigcap_{\delta<\lambda} A_\delta$ is also a club.

2) Suppose κ is a regular uncountable cardinal and $f : \kappa \to \kappa \times \kappa$ is a bijection. Show that $\{\alpha \in \kappa : f[\alpha] = \alpha \times \alpha\}$ is a club. (Here, $f[\alpha]$ denotes the pointwise image of α.)

3) For subsets $A, B \subseteq \omega$, we say that A is *almost contained* in B, and write $A \subseteq_* B$, if $A \setminus B$ is finite.

   The relation $\subseteq_*$ induces an equivalence relation $\approx$ on $\mathcal{P}(\omega)$ defined by the rule $A \approx B$ if $A \subseteq_* B$ and $B \subseteq_* A$ (i.e. $A \approx B$ if the symmetric difference of A and B is finite).

   We write $A \subsetneq_* B$ if $A \subseteq_* B$ and $B \not\approx A$ (i.e. $B \setminus A$ is infinite).

   i. Suppose that $\{A_n : n \in \omega\}$ is a $\subsetneq_*$-increasing chain of subsets of ω. Show that the chain does not have a $\subseteq_*$-least upper bound. That is, show that if B is a subset of ω such that $A_n \subseteq_* B$ for every n, then there is $C \subseteq \omega$ such that $A_n \subseteq_* C$ for every n, and $C \subsetneq_* B$.

   $$A_0 \subsetneq_* A_1 \subsetneq_* \ldots \subsetneq_* C \subsetneq_* B$$

   ii. Suppose that $\{A_n : n \in \omega\}$ is a $\subsetneq_*$-increasing chain of subsets of ω, and $\{B_n : n \in \omega\}$ is a $\subsetneq_*$-decreasing chain of subsets of ω. Suppose further that for every $n, m \in \omega$ we have $A_n \subseteq_* B_m$. Show that there is $C \subseteq \omega$ such that for every $n, m \in \omega$ we have $A_n \subsetneq_* C \subsetneq_* B_m$.

   $$A_0 \subsetneq_* A_1 \subsetneq_* \ldots \subsetneq_* C \subsetneq_* \ldots \subsetneq_* B_1 \subsetneq_* B_0$$

   Some culture: Part (ii.) hints at a famous theorem of Hausdorff, who showed in contrast that there is an ω_1-length $\subsetneq_*$-increasing chain $\{A_\delta : \delta \in \omega_1\}$ of subsets of ω lying below an ω_1-length $\subsetneq_*$-decreasing chain $\{B_\delta : \delta \in \omega_1\}$ such that no $C \subseteq \omega$ satisfies $A_\delta \subsetneq_* C \subsetneq_* B_\gamma$ for every $\delta, \gamma \in \omega_1$. Such a pair of chains is called a *Hausdorff gap*.