Ma 116a Homework #2

Due Thursday, February 1st at 1:00pm

1) An ordinal α is called *indecomposable* if for every $\beta < \alpha$ we have $\beta + \alpha = \alpha$. Prove that α is indecomposable if and only if $\alpha = \omega^\delta$ for some ordinal δ.

   Some culture: From (1) we can prove that every nonzero ordinal α has a unique decomposition as a finite sum of indecomposable ordinals $\alpha = \omega^{\delta_0} \cdot n_0 + \omega^{\delta_1} \cdot n_1 + \ldots + \omega^{\delta_k} \cdot n_k$ in which the exponents δ_i are strictly decreasing in i. The proof is as follows. Let δ_0 be maximal such that $\omega^{\delta_0} \leq \alpha$. Such a δ_0 exists since ordinal exponentiation is continuous in the exponent. Then it must be that there is a maximal $n_0 \in \omega$ such that $\omega^{\delta_0} \cdot n_0 \leq \alpha$, since otherwise we would have $\omega^{\delta_0+1} \leq \alpha$, contradicting the maximality of δ_0. Let α_1 be the unique ordinal such that $\alpha = \omega^{\delta_0} \cdot n_0 + \alpha_1$. If $\alpha_1 = 0$, we have found our decomposition. If not, we let δ_1 be maximal such that $\omega^{\delta_1} \leq \alpha_1$. Observe must have $\alpha_1 < \alpha_0$. As before, there must be some maximal $n_1 \in \omega$ such that $\omega^{\delta_1} \cdot n_1 \leq \alpha_1$, and we write $\alpha_1 = \omega^{\delta_1} \cdot n_1 + \alpha_2$. And so on. Since $\omega^{\delta_0} > \omega^{\delta_1} > \ldots$ is a decreasing sequence of ordinals, and there are no infinite decreasing sequences of ordinals, this process must terminate after a finite number of steps.

   The decomposition is called the *Cantor normal form* for α.

2) Prove that α is indecomposable if and only if for all suborders $A \subseteq \alpha$, at least one of A and $\alpha \setminus A$ is isomorphic to α.

3) A class function on the ordinals $F : ON \to ON$ is called *normal* if it satisfies the following two conditions:

   - F is strictly increasing: for all ordinals α, β, we have $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$,
   - F is continuous: for all limit ordinals α, we have $F(\alpha) = \sup\{F(\gamma) : \gamma < \alpha\}$.

   Prove that if F is normal, then F has arbitrarily large fixed points. That is, for every β there is $\alpha \geq \beta$ such that $F(\alpha) = \alpha$. Conclude that there are arbitrarily large ordinals α such that $\alpha = \omega^\alpha$.

4) The version of the Axiom of Choice that we stated in class is usually called the Well-Ordering Principle. Prove that the Well-Ordering Principle is equivalent to the standard formulation of AC:

   Axiom 9 (AC). For every family X of non-empty sets, there is a function $f : X \to \bigcup X$ such that for all $Y \in X$, $f(Y) \in Y$.

   This says that if X is a family of non-empty sets, there is a function f that picks out an element $f(Y)$ from each set Y in the family. Such a function is called a *choice function*.

5) Without using AC, prove the Cantor-Schroeder-Bernstein (CSB) theorem. That is, prove that if A and B are sets and there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection between A and B.

   Hint: Let $A_0 = A$, $B_0 = B$, $A_{n+1} = g[B_n]$, $B_{n+1} = f[A_n]$, $A_\infty = \bigcap_n A_n$, and $B_\infty = \bigcap_n B_n$. Let $h(x) = f(x)$ if $x \in A_\infty \cup \bigcup_n (A_{2n} \setminus A_{2n+1})$, otherwise let $h(x) = g^{-1}(x)$. Show that h is a bijection.

6) Fill in the details of the following alternate proof of the CSB theorem.

   - Claim: Suppose A', B, A are sets such that $A' \subseteq B \subseteq A$, and there is a bijection $f : A \to A'$. Then there is a bijection from A to B.

     Proof:

     – Let $Q = B \setminus f[A]$. Let $\mathcal{T} = \{X \subseteq A : Q \cup f[X] \subseteq X\}$, and let $T = \bigcap \mathcal{T}$.
     – Show that $T \in \mathcal{T}$, and conclude $Q \cup f[T] \subseteq T$.
     – Show that in fact $Q \cup f[T] = T$, and conclude $B = T \cup (f[A] \setminus f[T])$.
     – Conclude the proof.

- Use the claim to prove CSB.

Some culture: The previous proof of CSB implicitly relies on induction and recursion on the natural numbers. This proof, due to Zermelo, makes no mention of the natural numbers, much less induction or recursion!