Ma 116a Homework #1

Due Thursday, January 18th at 1:00pm

1) For sets x and y, let $\langle x, y\rangle$ denote the Kuratowski pair $\{\{x\}, \{x, y\}\}$ of x and y.

   i. Write a first-order formula $\varphi(z, x, y)$ in the language of set theory (i.e. using only the relations $\in$ and $=$) expressing $z = \langle x, y\rangle$.

   ii. Verify the claim made in class: for all sets x, y, z, w we have $\langle x, y\rangle = \langle z, w\rangle$ iff $x = z$ and $y = w$.

   iii. Show that we could not prove the claim in (ii.) if we had defined $\langle x, y\rangle$ to be $\{x, \{y\}\}$.

2) Let $\mathcal{M}$ be the universe of sets consisting of the empty set and a set $x = \{x\}$ that contains itself as a singleton. More formally, $\mathcal{M} = (\{\emptyset, x\}, \in)$, where $\in$ is the binary relation $\{(x, x)\}$ (that is, $\in$ asserts $x \in x$ and nothing else). Argue that $\mathcal{M}$ satisfies Set Existence, Extensionality, and Comprehension. Clearly, $\mathcal{M}$ does not satisfy Pairing. What about Union? Replacement?

3) Verify the following facts about ordinals that we stated in class:

   i. If x is an ordinal and $y \in x$, then y is an ordinal and $y = \text{pred}(y)$.

   ii. If x and y are ordinals and $x \cong y$ (as orders) then $x = y$ (as sets).

   iii. (Comparison for ordinals) If x and y are ordinals, then exactly one of the following holds: $x \in y$, $y \in x$, $x = y$.

   iv. If x, y, and z are ordinals and $x \in y$ and $y \in z$, then $x \in z$.

   v. If C is a non-empty set of ordinals, then $\exists x \in C \forall y \in C (x \in y \vee x = y)$.

4) Suppose α, β, and γ are ordinals.

   i. Suppose $\alpha < \beta$. Show that $\gamma + \alpha < \gamma + \beta$.

   ii. Suppose $\alpha \leq \beta$. Show there is a unique ordinal δ such that $\alpha + \delta = \beta$.

   iii. Show that if there is an order-preserving injection $f : \alpha \to \beta$ (not necessarily onto an initial segment of β), then $\alpha \leq \beta$.

   iv. Suppose $\alpha < \beta$. Show that $\alpha + \gamma \leq \beta + \gamma$. Find an example to show that $\alpha + \gamma < \beta + \gamma$ may fail to hold.