- Consider theory $T' = T \cup \{\exists \vec{v}\, (\varphi_1(\vec{v}) \wedge \varphi_2(\vec{v}) \wedge \dots \wedge \varphi_n(\vec{v}))\}$
- If T' had a model K, could interpret the constants $\vec{m}$ as the witnesses in K to the existential formula
- then have $K \models T \cup \Delta$
- thus no such model, i.e. $T \models \forall \vec{v}\, (\neg\varphi_1(\vec{v}) \vee \dots \vee \neg\varphi_n(\vec{v}))$
- this sentence is universal, hence in Σ
- But $M \models \exists \vec{v}\, (\varphi_1(\vec{v}) \wedge \dots \wedge \varphi_n(\vec{v}))$ and $M \models \Sigma$ contradiction ✓

- Thus there is $N \models T \cup \mathrm{Diag}(M)$
- Thus an embedding $j : M \to N$
- identifying M w/ $j[M]$ may assume $M \subseteq N$ (exercise: can you formalize?)
- Since $N \models T$ and T preserved under substructures, $M \models T$ ✓

Q4U: Which theories have "existential axiomatizations?"

<u>Chains of models</u>

Following prop'n generalizes observations like: if $G_1 \subseteq G_2 \subseteq \dots$ is an inc. chain of groups then $G = \bigcup G_i$ is a group.

<u>Def'n</u> Suppose $(I, <)$ is a linearly ordered index set. Sps for every $i \in I$, $\mathcal{M}_i$ is an $\mathcal{L}$-structure s.t. if $i<j$ then $\mathcal{M}_i \subseteq \mathcal{M}_j$. We say $(\mathcal{M}_i : i \in I)$ is a <u>chain of $\mathcal{L}$-structures</u>.

If moreover $\mathcal{M}_i \prec \mathcal{M}_j$ $\forall i<j$ say $(\mathcal{M}_i : i \in I)$ is an <u>elementary chain</u>.

- Given a chain of $\mathcal{L}$-structs $(\mathcal{M}_i : i \in I)$ can define an $\mathcal{L}$-struct $\mathcal{M}$ extending every $\mathcal{M}_i$.
- universe of $\mathcal{M}$ is $M = \bigcup M_i$
- given constant $c \in \mathcal{L}$, must have $c_i^{\mathcal{M}} = c_j^{\mathcal{M}}$ for all i,j, so define $c^{\mathcal{M}} = c_i^{\mathcal{M}}$ for any i
- given n-ary $R \in \mathcal{L}$ and $a_1, \dots, a_n \in M$ must exist i s.t. $a_1, \dots, a_n \in M_i$
- define $R^{\mathcal{M}}(a_1, \dots, a_n)$ iff $R^{\mathcal{M}_i}(a_1, \dots, a_n)$
- since $R^{\mathcal{M}_i} = R^{\mathcal{M}_j} | M_i$ for all $j > i$, this is well-defined
- likewise given n-ary $f \in \mathcal{L}$ and $a_1, \dots, a_n$ find i s.t. $a_1, \dots, a_n \in M_i$
- then $\exists b \in M_i$ so $f^{\mathcal{M}_i}(a_1, \dots, a_n) = b$
- define $f^{\mathcal{M}}(a_1, \dots, a_n) = b$
- well-defined since $f^{\mathcal{M}_i} = f^{\mathcal{M}_j} | M_i$ for $j > i$.

↪ this defines $\mathcal{L}$-struct $\mathcal{M}$.
↪ Clearly $\mathcal{M}_i \subseteq \mathcal{M}$ for all i.

<u>Prop'n</u> if $(M_i : i \in I)$ is an elementary chain, then the structure $M = \bigcup_{i \in I} M_i$ is an <u>elementary</u> extension of every M_i.

<u>PF</u>: Check for every ~~fmla~~ fmla $\varphi(\vec{v})$, every i, and every $\vec{a} \in M_i$ that
$$M_i \models \varphi(\vec{a}) \text{ iff } M \models \varphi(\vec{a})$$
by induction on $\varphi(\vec{v})$

(exercise...)

<u>Completeness</u>

<u>Def'n</u> An $\mathcal{L}$-theory T is <u>complete</u> if for every $\mathcal{L}$-sentence φ, either $T \models \varphi$ or $T \models \neg\varphi$

$\rightsquigarrow$ non-trivial case is when T is consistent, then exactly one of $T \models \varphi$, $T \models \neg\varphi$ holds, for every φ.

<u>Ex</u>: given $\mathcal{L}$-struct M, $T = Th(M)$ is complete. (...why?) In this case, exactly one of $\varphi \in T$, $\neg\varphi \in T$ holds.

<u>Problem</u> given M, can we find a (simple, maybe even finite) axiomatization

for Th($\mathcal{M}$)? i.e. a $T \subseteq$ Th($\mathcal{M}$) s.t.
$T \models \varphi$ iff Th($\mathcal{M}$) $\models \varphi$, for all φ.

<u>Notice</u>: given $T \subseteq$ Th($\mathcal{M}$), T axiomatizes Th($\mathcal{M}$) iff T is complete! (Why?)

So problem points to another: can we find methods to tell if a given T is complete?

Back-and-forth

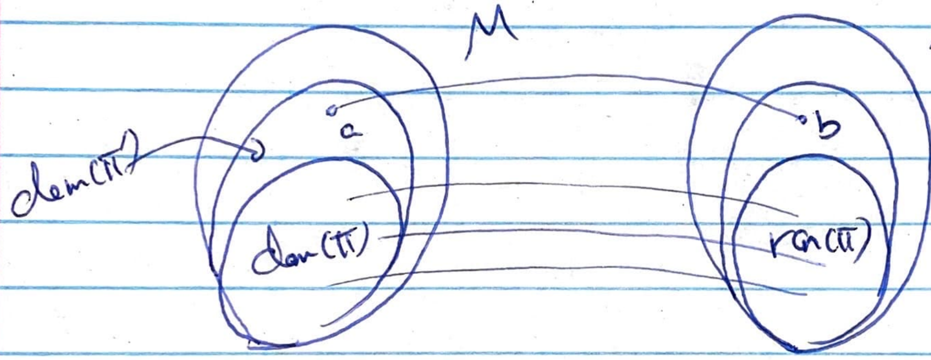
<u>Def'n</u> Given $\mathcal{M}, \mathcal{N}$ $\mathcal{L}$-structures
an injective map π with dom(π) $\subseteq M$, ran(π) $\subseteq N$ is a <u>partial isomorphism</u> if:
for all $c, R, f \in \mathcal{L}$:
if $c^{\mathcal{M}} \in$ dom(π) then $\pi(c^{\mathcal{M}}) = c^{\mathcal{N}}$
if $a_1, \ldots, a_n \in$ dom(π) then $R^{\mathcal{M}}(\vec{a})$ iff $R^{\mathcal{N}}(\pi(\vec{a}))$
if $\vec{a}, b \in$ dom(π) then $f^{\mathcal{M}}(\vec{a}) = b$ iff $f^{\mathcal{N}}(\pi(\vec{a})) = \pi(b)$

$\hookrightarrow$ will still write $\pi : M \to N$ even if π partial

<u>Ex's</u> - if $\pi : \mathcal{M} \to \mathcal{N}$ is an iso, then it's a partial iso
- empty map $\emptyset : \mathcal{M} \to \mathcal{N}$ is a partial iso (trivially)

<u>Def'n</u> a family I of partial iso's from $\mathcal{M}$ to $\mathcal{N}$ has the <u>back-and-forth property</u> if: ① for all $\pi \in I$ and $a \in M$
there is $\pi' \in I$ w/ $\pi' \supseteq \pi$ and $\mathrm{dom}(\pi') \ni a$
② for all $\pi \in I$ and $b \in N$
there is $\pi' \in I$ w/ $\pi' \supseteq \pi$ and $b \in \mathrm{ran}(\pi')$

↪ if there is such an I, write: $\mathcal{M} \cong_p \mathcal{N}$.
Say: $\mathcal{M}, \mathcal{N}$ are <u>partially</u> <u>isomorphic</u>.



- if $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \cong_p \mathcal{N}$: take $I = \{\pi\}$ where π is an isomorphism
- can have $\mathcal{M} \cong_p \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$:

- Let DLO denote axioms for linear orders + the density axiom: $\forall x \forall y\, (x<y \Rightarrow \exists z (x<z<y))$
- Let DLO* be DLO + the "no endpoints" axiom: $\forall x\, \exists y \exists z\, (y<x<z)$

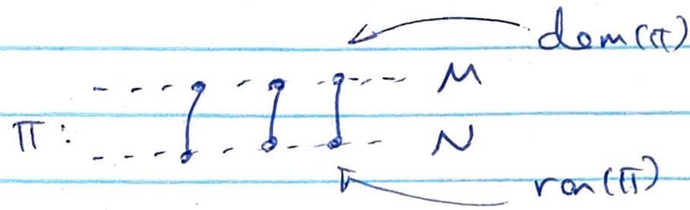
<u>Prop'n</u> if $\mathcal{M}, \mathcal{N} \models \mathrm{DLO}^*$, then $\mathcal{M} \cong_p \mathcal{N}$

Pf: - Let I = collection of all finite partial iso's from M to N ← dom/ran finite

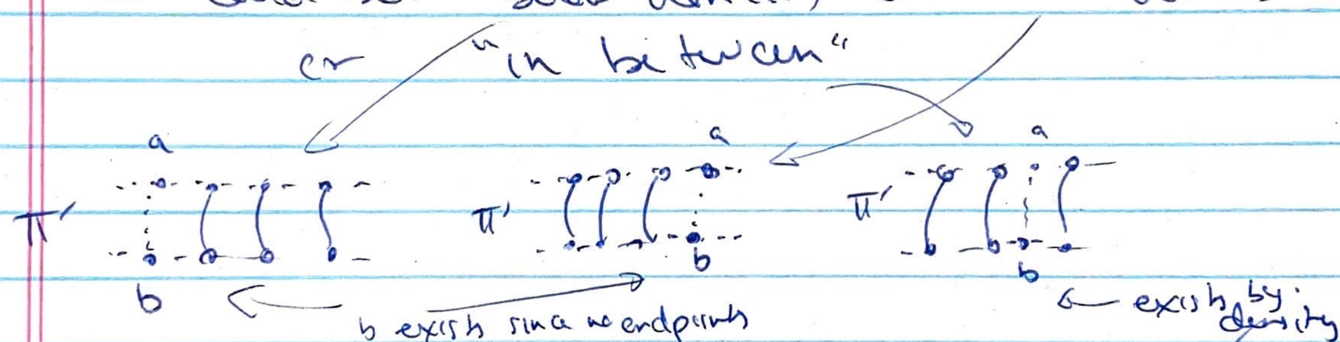
- Claim I has b-f prop.

① given $\pi \in I$ and $a \in M$, use either density or no-endpoints to find $b \in N$ s.t. $\pi' = \pi \cup \{(a,b)\}$ is order-preserving.

Pic:



a could be: "below dom(π)" or "above dom(π)" or "in between"



② symmetric for $\pi \in I$, $b \in N$. ✓

↳ since $(\mathbb{Q},<)$, $(\mathbb{R},<) \models DLO^*$, have $(\mathbb{Q},<) \equiv_p (\mathbb{R},<)$

↳ but of $(\mathbb{Q},<) \not\cong (\mathbb{R},<)$

$\mathbb{Q}$ is ctbl, $\mathbb{R}$ is not!

Prop'n If $\mathcal{M}, \mathcal{N}$ are countable and $\mathcal{M} \equiv_p \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.

Pf: - Sps I is a family of partial iso's w/ b-f property
- fix $\pi \in I$
- Enumerate $M = \{m_0, m_1, \ldots\}$ $N = \{n_0, n_1, \ldots\}$
- we build a sequence of partial iso's
$$\pi = \pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \ldots$$
s.t. for every i:
$m_i \in dom(\pi_{2i+1})$ (so: $m_i \in dom(\pi_j)$ $\forall j \geq 2i+1$)
$n_i \in dom(\pi_{2i+2})$

This is easy, using b-f prop:
- Stage $2i+1$: given π_{2i} pick $\pi_{2i+1} \in I$ w/ $\pi_{2i+1} \supseteq \pi_{2i}$ and $m_i \in dom(\pi_{2i+1})$
- Stage $2i+2$: given π_{2i+1} pick $\pi_{2i+2} \in I$ w/ $\pi_{2i+2} \supseteq \pi_{2i+1}$ and $n_i \in ran(\pi_{2i+2})$

- Let $\tilde{\pi} = \bigcup_i \pi_i$
- then $dom(\tilde{\pi}) = M$, $ran(\tilde{\pi}) = N$

- $\tilde{\pi}$ is an isomorphism! Why: so: $\mathcal{M} \cong \mathcal{N}$ as desired
if $\tilde{\pi}$ were not injective, or $\tilde{\pi}$ didn't preserve a c or f or R symbol, then could find a π_k that did the same, contradicting π_k is partial isomorphism ✓

Corollary (Cantor) any two <u>ctbl</u> models $\mathcal{M}, \mathcal{N} \models DLO^*$ are isomorphic

$\hookrightarrow$ since $(\mathbb{Q}, <) \models DLO^*$, this says: <u>every</u> ctbl dense linear order w/o endpoints is iso to $\mathbb{Q}$!

Theorem: if $\mathcal{M} \equiv_p \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Pf: <u>Recall</u>: we showed before: if $\mathcal{M} \cong \mathcal{N}$ then ~~[illegible]~~ $\mathcal{M} \equiv \mathcal{N}$ by showing that if $\pi: \mathcal{M} \to \mathcal{N}$ an iso then for any $\varphi(\vec{v})$ and $\vec{a} \in \mathcal{M}$ we have

$$\mathcal{M} \models \varphi(\vec{a}) \text{ iff } \mathcal{N} \models \varphi(\pi(\vec{a}))$$

- Sps I is a family of partial iso's witnessing $\mathcal{M} \equiv_p \mathcal{N}$
- we show: for any $\varphi(\vec{v})$, that for any ~~[illegible]~~ $\pi \in I$ and ~~[illegible]~~ $a_1, \dots, a_n \in \mathrm{dom}(\pi)$ we have:

$$(*) \qquad \mathcal{M} \models \varphi(\vec{a}) \text{ iff } \mathcal{N} \models \varphi(\pi(\vec{a}))$$

by induction on φ.
- then we're done, since if φ a sentence we get $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.

(Base Case) Sps φ atomic, fix $\pi \in I$.
Re-run atomic proof for $\mathcal{M} \cong \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$
using π instead of $\tilde{\pi}$.

(Induction Step) if ⊛ true for φ, ψ
then ~~...~~ easy to check true
for $\neg\varphi$ and $\varphi \wedge \psi$.

- Interesting case is: Sps ⊛ true for $\varphi(\vec{v}, w)$ and consider $\exists w\, \varphi(\vec{v}, w)$
- fix $\pi \in I$ and $a_1, \ldots, a_n \in \text{dom}(\pi)$
We prove ⊛ for $\exists w\, \varphi(\vec{v}, w)$

($\Rightarrow$) if $\mathcal{M} \models \exists w\, \varphi(\vec{a}, w)$ there is $b \in M$
s.t. $\mathcal{M} \models \varphi(\vec{a}, b)$
- using b-f prop find $\pi' \supseteq \pi$ w/ $\pi' \in I$
and $b \in \text{dom}(\pi')$
- by ind. hyp. $\mathcal{N} \models \varphi(\pi'(\vec{a}), \pi'(b))$ so
$\mathcal{N} \models \exists w\, \varphi(\pi'(\vec{a}), w)$, i.e. $\mathcal{N} \models \exists w\, \varphi(\pi(\vec{a}), w)$ ✓

($\Leftarrow$) symmetric ✓

Corollary: if $\mathcal{M} \models DLO^*$ then $\mathcal{M} \equiv (\mathbb{Q}, <)$
so $Th(\mathcal{M}) = Th(\mathbb{Q})$
→ so: DLO^* is complete!
→ more specifically: DLO^* axiomatizes $Th(\mathbb{Q}, <)$