Claim: if $\mathcal{M} \models T_i$, we can interpret the function symbols in $\mathcal{L}_{i+1} \setminus \mathcal{L}_i$ in $\mathcal{M}$ so that $\mathcal{M} \models T_{i+1}$

Pf: - Sps $\varphi(\vec{v}, w)$ an $\mathcal{L}_i$-fmla ($\vec{v} = v_1, \ldots, v_n$)
- we define a function $g: M^n \to M$
- given $\vec{a} = (a_1, \ldots, a_n) \in M^n$
(1) if there is $b \in M$ s.t. $\mathcal{M} \models \varphi(\vec{a}, b)$ pick one such b (axiom of choice!) and set $g(\vec{a}) = b$
(2) if no such b, pick any $c \in M$, let $g(\vec{a}) = c$

- by construction: if $\mathcal{M} \models \exists w\, \varphi(\vec{a}, w)$ then $\mathcal{M} \models \varphi(\vec{a}, g(\vec{a}))$
- if we interpret f_φ as g, have $\mathcal{M} \models \Phi_\varphi$
- if we do this for all $\mathcal{L}_i$-fmlas φ, get $\mathcal{M} \models T_{i+1}$ ✓

(Abusing notation here, really this $\mathcal{M}$ is expansion of the original)

- Now: iterate claim to interpret all symbols $f \in \mathcal{L}^* \setminus \mathcal{L}$
- let $\mathcal{M}^*$ denote resulting expansion of $\mathcal{M}$.
- for any $\varphi(\vec{v}, w)$ an $\mathcal{L}^*$-fmla, there is i s.t. φ is $\mathcal{L}_i$-fmla
- so then $\Phi_\varphi \in T_{i+1} \subseteq T^*$, so T^* has ~~[illegible]~~ bis f.

- Moreover $M^* \models T^*$ by construction ✓

- To gain int'n for Skolem functions consider following Q: given M and subset → $X \subseteq M$, what is smallest substructure → substructure $M' \subseteq M$ s.t. $X \subseteq M'$?

Answer: Get M' by adding any constants c^M not already there, and then closing under all functions f^M.

Idea: if we have built in skolem functions, then M' will actually be elementary.

Theorem (Downward LS) Sps M is $\mathcal{L}$-structure and $X \subseteq M$ (important case: $X = \emptyset$).
There is $N \prec M$ an elem substructure s.t. $X \subseteq N$ and $|N| \le \max\{|X|, |\mathcal{L}|, \aleph_0\}$
"as small as possible"

Pf: - by lemma may assume $Th(M)$ has bisf (if not, apply lemma - doesn't increase size of $\mathcal{L}$ if already infinite)
- We build an increasing sequence of subsets $X_0 \subseteq X_1 \subseteq \ldots$

- let $X_0 = X \cup \{c^{\mathcal{M}} : c$ a constant-symbol$\}$
- given X_i, let $X_{i+1} = X_i$ + all outputs of functions $f^{\mathcal{M}}$ w/ inputs from X_i, i.e.
  $X_{i+1} = X_i \cup \{f^{\mathcal{M}}(\vec{a}) : f$ a function symbol, $\vec{a} \in X_i\}$
- let $N = \bigcup X_i$
- then $|N| \leq \max\{|X|, |L|, \aleph_0\}$

- by construction, for every function symbol f, N is closed under $f^{\mathcal{M}}$
- so can define $f^{\mathcal{N}} = f^{\mathcal{M}} \restriction N$
- let $R^{\mathcal{N}} = R^{\mathcal{M}} \restriction N$ for all $R \in \mathcal{L}$
- $c^{\mathcal{N}} = c^{\mathcal{M}}$ for all $c \in \mathcal{L}$
  ← belongs to $\mathcal{N}$

- then $\mathcal{N} \subseteq \mathcal{M}$ is a substructure of $\mathcal{M}$.
- If $\varphi(\vec{v}, w)$ is a fmla and $\mathcal{M} \models \varphi(\vec{a}, b)$ for some $\vec{a} \in \mathcal{N}$ and $b \in \mathcal{M}$ then
  $\mathcal{M} \models \varphi(\vec{a}, f_\varphi(\vec{a}))$

- but $f_\varphi(\vec{a}) \in N$ since $f_\varphi = f^{\mathcal{M}}$ for some function symbol f
- By Tarski-Vaught test, $\mathcal{N} \prec \mathcal{M}$ ✓

Ex's

① (Upward LS) Let $\mathcal{N} = (\mathbb{N}, +, \cdot, <, 0, 1)$
There are elementary extension $\mathcal{N}_\kappa$ of $\mathcal{N}$ for any infinite cardinal κ.

② (Downward LS) Let $\mathcal{M} = (\mathbb{R}, <)$
There is a countable elementary substructure $\mathcal{N} \prec \mathcal{M}$
(In fact $\mathcal{N} = (\mathbb{Q}, <)$ is such a structure, but still need a little more to see this.)

Note: neither upward/downward LS tell us what the elem extensions/submodels of a given $\mathcal{M}$ look like
... just that they can be "constructed" abstractly by compactness/iteratively adding witnesses to all existential statements.

③ Let $\mathcal{M} = (\mathbb{C}, +, \cdot, 0, 1)$
- By LS, there is a ctbl elem substruct $(\overline{\mathbb{Q}}, +, \cdot, 0, 1) \prec \mathcal{M}$ s.t. $\mathbb{Q} \subseteq \overline{\mathbb{Q}}$
- Notice: by elementarity, $\overline{\mathbb{Q}}$ is algebraically closed, i.e. if
$a_n x^n + \dots + a_1 x + a_0$ is a poly w/

in part. if $a_i \in \mathbb{Q}$

coeffs $a_i \in \hat{\mathbb{Q}}$, then has a root in $\hat{\mathbb{Q}}$ (70)

Why: $\exists x\,(a_n x^n + \dots + a_1 x + a_0 = 0)$ true in $\mathbb{C}$ since alg. closed, hence true in $\hat{\mathbb{Q}}$

- So e.g. $\hat{\mathbb{Q}}$ contains $\sqrt{2}, \sqrt{3}, i, \dots$ any other complex number that is root of a poly w/ coeffs in $\mathbb{Q}$.

Universal axiomatizations

Def'n An $\mathcal{L}$-theory T has a universal axiomatization if there is an $\mathcal{L}$-theory T' consisting of universal sentences s.t. $\mathcal{M} \models T$ iff $\mathcal{M} \models T'$, or equiv:

$T \models \sigma$ iff $T' \models \sigma$ for all sentences σ

Def'n A universal sentence is one of the form $\forall \vec{v}\, \varphi(\vec{v})$ where $\varphi(\vec{v})$ is quantifier free

e.g: $\forall x \forall y \forall z\, (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$

Theorem an $\mathcal{L}$-theory T has a universal axiomatization if and only if T is preserved under substructure, i.e. whenever $\mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{N} \models T$.

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Pf: ($\Rightarrow$) - Sps T has univ. ax'm'n T',
Sps $\mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$
- Fix $\forall \vec{v}\, \varphi(\vec{v})$ from T'
- since $\varphi(\vec{v})$ is quant. free, knew from before $\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{N} \models \varphi(\vec{a})$ for all $\vec{a} \in M$
- thus $\mathcal{N} \models \forall \vec{v}\, \varphi(\vec{v})$ implies $\mathcal{M} \models \forall \vec{v}\, \varphi(\vec{v})$
- So $\mathcal{M} \models T'$, so $\mathcal{M} \models T$

($\Leftarrow$) - Sps T preserved under substructures
- Let $\Sigma = \{\varphi : \varphi$ a univ. sentence and $T \models \varphi\}$
↳ WTS: Σ is univ. ax'm'n of T
- Fix $\mathcal{M}$
- if $\mathcal{M} \models T$ then $\mathcal{M} \models \Sigma$ by defn of Σ
- We show: if $\mathcal{M} \models \Sigma$ then $\mathcal{M} \models T$
- to use hypothesis, we find $\mathcal{N} \supseteq \mathcal{M}$ s.t. $\mathcal{N} \models T$

<u>Claim</u> $T \cup \mathrm{Diag}(\mathcal{M})$ is satisfiable
<u>Pf</u>: - if not, by compactness there is a finite $\Delta \subseteq \mathrm{Diag}(\mathcal{M})$ s.t. $T \cup \Delta$ is unsat.
- Say $\Delta = \{\psi_1, \dots, \psi_n\}$
- let $\vec{m}$ denote the constants (corresponding to el'ts of $\mathcal{M}$) that appear in the ψ_i
- each ψ_i of form $\varphi_i(\vec{m})$ where φ_i is atomic/neg'n of atomic and $\mathcal{M} \models \varphi_i(\vec{m})$

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- Consider theory $T' = T \cup \{\exists \vec{v}(\varphi_1(\vec{v}) \wedge \varphi_2(\vec{v}) \wedge \dots \wedge \varphi_n(\vec{v}))\}$
- If T' had a model K, could interpret the constants $\vec{m}$ as the witnesses in K to the existential formula
- then have $K \models T \cup \Delta$
- thus no such model, i.e. $T \models \forall \vec{v}(\neg\varphi_1(\vec{v}) \vee \dots \vee \neg\varphi_n(\vec{v}))$
- this sentence is universal, hence in Σ
- But $M \models \exists \vec{v}(\varphi_1(\vec{v}) \wedge \dots \wedge \varphi_n(\vec{v}))$ and $M \models \Sigma$ contradiction ✓

- Thus there is $N \models T \cup Diag(M)$
- Thus an embedding $j : M \to N$
- identifying M w/ $j[M]$ may assume $M \subseteq N$ (exercise: can you formalize?)
- Since $N \models T$ and T preserved under substructures, $M \models T$ ✓

Q4U: Which theory have "existential axiomatizations"?

Chains of models

Following prop'n generalizes observations like: if $G_1 \subseteq G_2 \subseteq \dots$ is an inc. chain of groups then $G = \bigcup G_i$ is a group