$\mathcal{M}$ has a least el't 0
which is followed by next least el't 1
" " $1+1$
$\vdots$

$$\mathcal{M}: \quad \underset{0^{\mathcal{M}}}{\bullet} \quad \underset{1^{\mathcal{M}}}{\bullet} \quad \underset{1^{\mathcal{M}}+1^{\mathcal{M}}}{\bullet} \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$

What claim says is there is $\mathcal{M} \models Th(\mathbb{N})$ with an "infinite element" (!)

$$\mathcal{M}: \quad \underbrace{\bullet \; \bullet \; \bullet \; \cdots}_{\mathbb{N}} \qquad \underset{a}{\bullet}$$

Once there is one infinite element, there are many.

$$\mathcal{M}: \quad \underbrace{\bullet \; \bullet \; \bullet \; \cdots}_{\mathbb{N}} \qquad \cdots \; \underset{a-1}{\bullet} \; \underset{a}{\bullet} \; \underset{a+1}{\bullet} \; \bullet \; \cdots \qquad \cdots$$

So: even if we take our "axioms" to be all first-order statements which are true of $(\mathbb{N}, +, \cdot, <, 0, 1)$ .. these axioms not enough to characterize $\mathbb{N}$!

Pf of claim: - Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where
c is a new constant symbol
- Let φ_n be $\underbrace{1+1+\dots+1}_{n\text{-times}} < c$

- Let $T' = Th(\mathcal{N}) \cup \{\varphi_1, \varphi_2, \dots\}$
- If $\Delta \subseteq T'$ is finite, then
$\Delta \subseteq Th(\mathcal{N}) \cup \{\varphi_1, \dots, \varphi_N\}$ for some large enough N.
- Let $\mathcal{M}$ be $\mathcal{L}^*$ expansion of $\mathcal{N}$ where we let $c^{\mathcal{M}} = N+1$
- Then $\mathcal{M} \models \Delta$
- Thus T' is fin sat.
$\Rightarrow$ there is $\mathcal{M} \models T'$ by compactness
- In $\mathcal{M}$, $c^{\mathcal{M}} > n = 1+1+\dots+1$ for all n ✓

Lowenheim-Skolem Theorem

Recall: if $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structs
write $\mathcal{M} \subseteq \mathcal{N}$ ("$\mathcal{M}$ a substructure of $\mathcal{N}$")
if $M \subseteq N$, and $f^{\mathcal{M}} = f^{\mathcal{N}}|M$
$R^{\mathcal{M}} = R^{\mathcal{N}}|M$
$c^{\mathcal{M}} = c^{\mathcal{N}}$ for all f/R/c symbols

Equivalently: (why?)
$\mathcal{M} \subseteq \mathcal{N}$ if
$\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{N} \models \varphi(\vec{a})$
for all atomic fmlas φ and $\vec{a} \in M$

- write $\mathcal{M} \prec \mathcal{N}$ ("$\mathcal{M}$ is elementary substructure of $\mathcal{N}$", "$\mathcal{N}$ is elem extension of $\mathcal{M}$") if $M \subseteq N$ and $\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{N} \models \varphi(\vec{a})$ for all fmlas $\varphi(\vec{v})$ and $\vec{a} \in M$.

- an injective $j: M \to N$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$ if $j[\mathcal{M}] \subseteq \mathcal{N}$
- is an elementary embedding if $j[\mathcal{M}] \prec \mathcal{N}$, i.e. for all φ and $\vec{a} \in M$
$$\mathcal{M} \models \varphi(\vec{a}) \text{ iff } \mathcal{N} \models \varphi(j(\vec{a}))$$

- structures strongly resemble their elem substructures — if $\mathcal{M} \prec \mathcal{N}$ then e.g. every sentence φ true in $\mathcal{N}$ iff in $\mathcal{M}$
- yet we'll prove: given an infinite $\mathcal{M}$, we can find elem extensions of $\mathcal{M}$ as big as we want, and elem substructures that are as small as possible.

<u>Def'n</u> Sps $\mathcal{M}$ an $\mathcal{L}$-structure. Let $\mathcal{L}_M$ be lang. obtained by adding to $\mathcal{L}$ a constant symbol m for each el't of M.

(e.g. if $\mathcal{M} = (\mathbb{N}, +, \cdot, <, 0, 1)$ then $\mathcal{L}_M = \{+, \cdot, <, 0, 1, 2, 3, \ldots\}$)

The atomic diagram of $\mathcal{M}$ (write: $\mathrm{Diag}(\mathcal{M})$) is the theory $\{\varphi(m_1, \ldots, m_n) : \varphi$ an atomic or neg'n of atomic $\mathcal{L}$-fmla and $\mathcal{M} \models \varphi(m_1, \ldots, m_n)\}$

(viewing $\mathcal{M}$ as an $\mathcal{L}_M$-structure here, interpreting every new constant symbol as the el't it represents)

The elementary diagram (write: $\mathrm{Diag}_{el}(\mathcal{M})$) is $\{\varphi(m_1, \ldots, m_n) : \varphi(\vec{v})$ an $\mathcal{L}$-fmla and $\mathcal{M} \models \varphi(\vec{m})\}$

Lemma Sps $\mathcal{N}$ is an $\mathcal{L}_M$-structure
① if $\mathcal{N} \models \mathrm{Diag}(\mathcal{M})$, then (viewing $\mathcal{N}$ as an $\mathcal{L}$-structure) there is an embedding of $\mathcal{M}$ into $\mathcal{N}$
② if $\mathcal{N} \models \mathrm{Diag}_{el}(\mathcal{M})$ then there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Pf: ① Define $j: \mathcal{M} \to \mathcal{N}$ by $j(m) = m^{\mathcal{N}}$
i.e. we send m to interp (in $\mathcal{N}$) of constant symbol for m.
- j is "clearly" an embedding. Explicitly:
- given distinct $m_1, m_2 \in M$, we have $m_1 \neq m_2 \in \mathrm{Diag}(\mathcal{M})$
- thus $j(m_1) = m_1^{\mathcal{N}} \neq m_2^{\mathcal{N}} = j(m_2)$

- so j is injective
- given constant symbol $c \in \mathcal{L}$ have $c^{\mathcal{M}} = m$ for some $m \in \mathcal{M}$, so $c = m \in \text{Diag}(\mathcal{M})$, so $j(c^{\mathcal{M}}) = j(m) \overset{?}{=} m^{\mathcal{N}} = c^{\mathcal{N}}$
- given $R \in \mathcal{L}$, if $R^{\mathcal{M}}(m_1, \ldots, m_n)$ then $R(m_1, \ldots, m_n) \in \text{Diag}(\mathcal{M})$, so $R^{\mathcal{N}}(j(m_1), \ldots, j(m_n))$
- given $f \in \mathcal{L}$, if $f^{\mathcal{M}}(m_1, \ldots, m_n) = m$ then $f(m_1, \ldots, m_n) = m \in \text{Diag}(\mathcal{M})$ so $f^{\mathcal{N}}(j(m_1), \ldots, j(m_n)) = j(m)$
  $\Rightarrow j$ an embedding ✓

② observe in this case the map j is elementary. ✓

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if $\mathcal{M}$ is an infinite struct. then lemma + compactness allows us to build elem extensions of $\mathcal{M}$ that are as big as we want.

<u>Theorem</u> (Upward Lowenheim-Skolem)
Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and fix a cardinal $\kappa \geq |M|$. Then there exists an $\mathcal{L}$-structure $\mathcal{N}$ s.t. $|N| = \kappa$ and there is an elementary embedding $j : \mathcal{M} \to \mathcal{N}$.

Pf: - fix $\kappa \geq |M|$
- we assume $|\mathcal{L}| \leq |M|$ so we can apply version of compactness we proved, but this is unnecessary if one examines proof
- introduce new constants $c_i : i < \kappa$
- for every $i \neq j$ add sentence $c_i \neq c_j$ to $Diag_{el}(\mathcal{M})$
- let $T = Diag_{el}(\mathcal{M}) \cup \{c_i \neq c_j : i, j < \kappa, i \neq j\}$ be resulting theory in lang $\mathcal{L}' = \mathcal{L} \cup \{c_i : i < \kappa\}$

- if $\Delta \subseteq T$ is finite, then contains only finitely many assertions of the form $c_i \neq c_j$.
- if we interpret symbols c_i appearing in Δ as arbitrary (but distinct) el'ts of $\mathcal{M}$ (is infinite!) get $\mathcal{M} \models \Delta$.
- so T is fin. sat. hence satisfiable by compactness.
- since $\kappa = \max\{|M|, |\mathcal{L}|\}$ there is $\mathcal{N} \models T$ of size $\leq \kappa$
- must have $|N| = \kappa$ since the $c_i^{\mathcal{N}}$'s are κ-many distinct el'ts of $\mathcal{N}$

- Since $\mathcal{N} \models Diag_{el}(\mathcal{M})$ there is $j : \mathcal{M} \to \mathcal{N}$ elementary ✓

- if we identify $\mathcal{M}$ w/ $j[\mathcal{M}]$, then says $\exists$ elem extension of $\mathcal{M}$ of every cardinality

- "Upward" = larger.
- Can we get smaller? Need following:

<u>Prop'n</u> (Tarski–Vaught test) Sps $\mathcal{M} \subseteq \mathcal{N}$.
Then $\mathcal{M} \prec \mathcal{N}$ iff we have:
(*) for any fmla $\varphi(w, \vec{v})$ and $\vec{a} \in \mathcal{M}$
if there is $b \in \mathcal{N}$ s.t. $\mathcal{N} \models \varphi(b, \vec{a})$ then can find $c \in \mathcal{M}$ s.t. $\mathcal{N} \models \varphi(c, \vec{a})$

"if there is a witness for φ in $\mathcal{N}$, can find a witness in $\mathcal{M}$"
But notice: witness ... from pov. of $\mathcal{N}$!

<u>Pf</u> ($\Rightarrow$) if $\mathcal{M} \prec \mathcal{N}$, then (*) follows immediately from elementarity
($\Leftarrow$) Assume (*).
<u>Must check</u> for every $\varphi(\vec{v})$ and $\vec{a} \in \mathcal{M}$:
$\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{N} \models \varphi(\vec{a})$

by induction on $\varphi(\vec{v})$
- if $\varphi(\vec{v})$ atomic, get this just from $\mathcal{M} \subseteq \mathcal{N}$
- if true for φ, also for $\neg\varphi$ since $\mathcal{M} \models \neg\varphi(\vec{a})$ iff $\mathcal{M} \not\models \varphi(\vec{a})$ iff (induction) $\mathcal{N} \not\models \varphi(\vec{a})$ iff $\mathcal{N} \models \neg\varphi(\vec{a})$
- if true for φ, χ then similarly easily true for $\varphi \wedge \chi$

– Sps claim true for $\varphi(w, \vec{v})$, consider $\exists w\, \varphi(w, \vec{v})$

– If $\mathcal{M} \models \exists w\, \varphi(w, \vec{a})$ then $\exists\, c \in \mathcal{M}$ s.t. $\mathcal{M} \models \varphi(c, \vec{a})$, thus $\mathcal{N} \models \varphi(c, \vec{a})$ by induction so $\mathcal{N} \models \exists w\, \varphi(w, \vec{a})$

– and if $\mathcal{N} \models \exists w\, \varphi(w, \vec{a})$ there is $b \in \mathcal{N}$ s.t. $\mathcal{N} \models \varphi(b, \vec{a})$ so by (*) there is $c \in \mathcal{M}$ s.t. $\mathcal{N} \models \varphi(c, \vec{a})$ so by induction $\mathcal{M} \models \varphi(c, \vec{a})$ and $\mathcal{M} \models \exists w\, \varphi(w, \vec{a})$

$\Rightarrow$ Claim holds for all $\varphi(\vec{v})$ and $\vec{a} \in M$ by induction ✓

Prop'n suggests idea:

Given $\mathcal{N}$, to find small-as-possible $\mathcal{M} \subset \mathcal{N}$

– find set of witnesses M_1 for all existential formulas $\exists w\, \varphi(w)$

– then witnesses M_2 for all $\exists w\, \varphi(w, \vec{a})$ with $\vec{a} \in M_1$

– then witnesses M_3 " $\exists w\, \varphi(w, \vec{a})$ " $\vec{a} \in M_2$

... etc.

then take union to get $\mathcal{M}$.

– This works! But instead: we introduce functions that will do this work for us

– somewhat analogous to witness property

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"bisf"

Def'n An $\mathcal{L}$-theory T has built-in Skolem functions if for every $\mathcal{L}$-fmla $\varphi(w,\vec{v})$ there is a symbol f s.t.

$$T \models \forall \vec{v}\,(\exists w\, \varphi(w,\vec{v}) \Rightarrow \varphi(f(\vec{v}),\vec{v}))$$

Lemma Sps T is an $\mathcal{L}$-theory. Then there is $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a $\mathcal{L}^*$-theory s.t. T^* has bisf and $|\mathcal{L}^*| = \max\{|\mathcal{L}|, \aleph_0\}$.
Moreover, if $\mathcal{M} \models T$ there is an $\mathcal{L}^*$ expansion $\mathcal{M}^* \models T^*$.

Pf: We build a sequence of langs
$\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$
and theories $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$
s.t. T_i is $\mathcal{L}_i$-theory

- Given $\mathcal{L}_i$, for every $\mathcal{L}_i$-fmla $\varphi(v_1,\dots,v_n,w)$ we introduce an n-ary f'n symbol f_φ
- Let $\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_\varphi : \varphi(\vec{v},w) \text{ an } \mathcal{L}_i\text{-fmla}\}$

reversing $\vec{v}$ and w here for no reason

x x

- for each $\varphi(\vec{v},w)$ an $\mathcal{L}_i$-fmla, let Φ_φ be the sentence
$$\forall \vec{v}\,(\exists w\, \varphi(\vec{v},w) \Rightarrow \varphi(\vec{v}, f_\varphi(\vec{v})))$$
- let $T_{i+1} = T_i \cup \{\Phi_\varphi\}$
- let $\mathcal{L}^* = \bigcup_i \mathcal{L}_i$, $T^* = \bigcup_i T_i$