From last time:

- Sps T an $\mathcal{L}$-theory
- T has witness property if for every fmla $\varphi(v)$ w/ one free var there is $c \in \mathcal{L}$ s.t. $T \models (\exists v\, \varphi(v)) \Rightarrow \varphi(c)$
- T is maximal if for every sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$
- we proved:

  Theorem if T is max'l, fin. sat. theory w/ witness property, then T has a model

Pf idea: - define $c \sim d$ for constants $c, d \in \mathcal{C}$ iff $c = d \in T$

- Let c^* denote equiv. class of c
- Define $\mathcal{M}$ by:

  $M = \mathcal{C}/\sim$

  $c^{\mathcal{M}} = c^*$ for all $c \in \mathcal{C}$

  $R^{\mathcal{M}}(c_1^*, \dots, c_n^*)$ iff $R(c_1, \dots, c_n) \in T$ for all rel'n symbols R

  $f^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$ iff $f(c_1, \dots, c_n) = d \in T$ " " function symbols f

- Check: $\mathcal{M} \models \varphi(\vec{c}^*)$ iff $\varphi(\vec{c}) \in T$

  $\Rightarrow \mathcal{M} \models T$

- We WTS compactness: if T is fin. sat. then T has a model
- strategy: check that T can be extended to a maximal theory w/ witness prop.

<u>Prop'n 1</u>: Sps T is a fin. sat. $\mathcal{L}$-theory. There is a lang $\mathcal{L}^* \supseteq \mathcal{L}$ and a fin. sat. $\mathcal{L}^*$-theory $T^* \supseteq T$ s.t. <u>any</u> $\mathcal{L}^*$-theory extending T^* has witness property.

<u>Pf idea</u>: get witness prop. iteratively:
- for each fmla $\varphi(v)$ add a constant c
- ... now there are new fmlas $\chi(v)$ involving these new constants
- add a constant for each of these
- etc...
- expand T along the way to make sure each c is a witness to corresponding $\varphi(v)$.

<u>Pf</u>: - for each $\mathcal{L}$-fmla $\varphi(v)$ w/ one free var. let c_φ be a new constant
- let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\varphi : \varphi(v)$ an $\mathcal{L}$-fmla$\}$
- for each $\mathcal{L}$-fmla $\varphi(v)$ let Θ_φ be sentence $(\exists v\, \varphi(v)) \Rightarrow \varphi(c_\varphi)$

- let $T_1 = T \cup \{\theta_\varphi : \varphi(v)$ an $\mathcal{L}$-fmla$\}$
- by construction: for every $\mathcal{L}$-fmla $\varphi(v)$ there is $c \in \mathcal{L}$, s.t. $T_1 \models (\exists v\, \varphi(v)) \Rightarrow \varphi(c)$

in fact, belongs to T_1

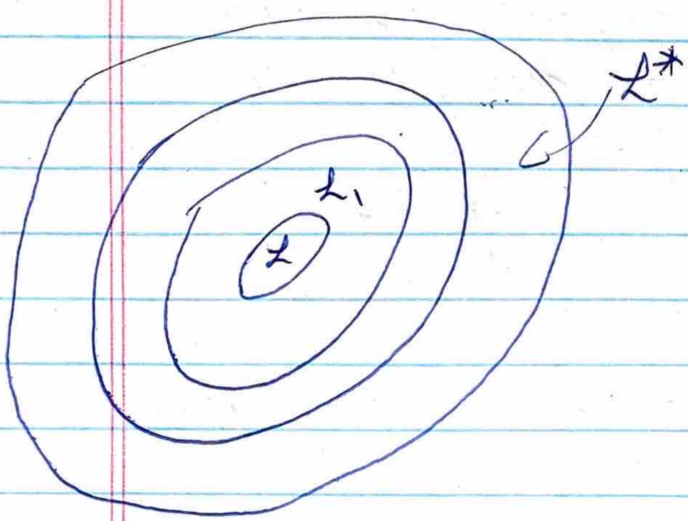
Claim T_1 is fin sat.

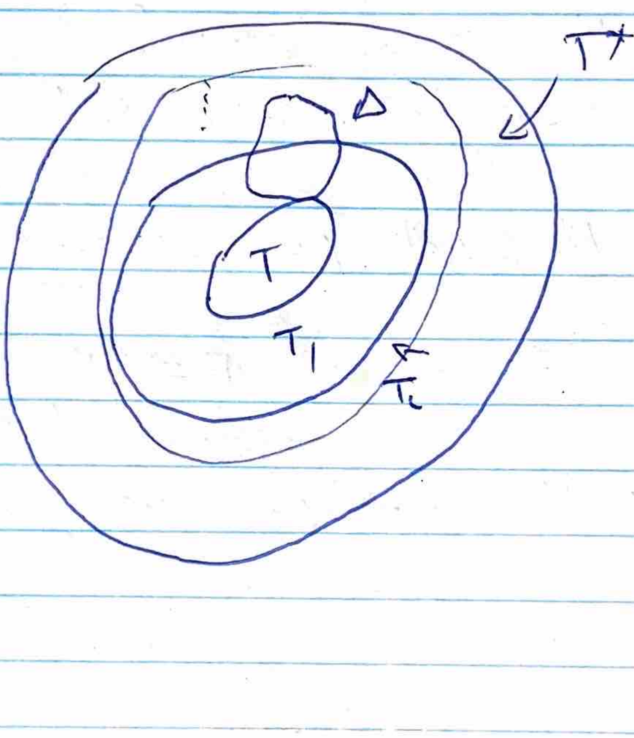
Pf - let $\Delta \subseteq T_1$ be finite
- then $\Delta = \Delta_0 \cup \{\theta_{\varphi_1}, \ldots, \theta_{\varphi_n}\}$ where $\Delta_0 \subseteq T$ and $\varphi_1(v), \ldots, \varphi_n(v)$ are $\mathcal{L}$-fmlas
- since T is fin sat there is $\mathcal{M} \models \Delta_0$
- we define an expansion $\mathcal{M}_1$ of $\mathcal{M}$ s.t. $\mathcal{M}_1 \models \Delta$
- $\mathcal{M}_1$ is same as $\mathcal{M}$, just need to interpret the new constant symbols appearing in Δ
- these are $c_{\varphi_1}, \ldots, c_{\varphi_n}$
- for a given $i \leq n$, if there is $a_i \in M$ s.t. $\mathcal{M} \models \varphi_i(a_i)$, then choose one such a_i and let $c_{\varphi_i}^{\mathcal{M}_1} = a_i$
- if no such a_i, choose any $b_i \in M$ and let $c_{\varphi_i}^{\mathcal{M}_1} = b_i$
- by construction, $\mathcal{M}_1 \models (\exists v\, \varphi_i(v)) \Rightarrow \varphi_i(c_i)$ i.e. $\mathcal{M}_1 \models \theta_{\varphi_i}$ for all $i \leq n$
- since $\mathcal{M}_1$ is an expansion of $\mathcal{M}$ also have $\mathcal{M}_1 \models \Delta_0$

so $\mathcal{M}_1 \models \Delta$ ✓

- Now we iterate above construction: get a sequence of langs $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$ and theories $T \subseteq T_1 \subseteq T_2 \subseteq \dots$
  s.t. for each i, T_{i+1} is fin. sat. $\mathcal{L}_{i+1}$ theory and for every $\mathcal{L}_i$-fmla $\varphi(v)$ w/ one free var, there is $c \in \mathcal{L}_{i+1}$ s.t. $T_{i+1} \models (\exists v\, \varphi(v)) \Rightarrow \varphi(c)$

- then: if we let $\mathcal{L}^* = \bigcup \mathcal{L}_i$, $T^* = \bigcup T_i$ then T^* is an $\mathcal{L}^*$-theory that, by construction, has witness prop. (why?)
- <u>and</u>: if $\Delta \subseteq T^*$ is finite then, since it's finite $\Delta \subseteq T_i$ for some i
- since T_i is fin. sat. there is $M \models \Delta$
- hence T^* is fin. sat. ✓





<u>Note</u>: if $|\mathcal{L}_i|$ denotes cardinality of $\mathcal{L}_i$
then $|\mathcal{L}_{i+1}| \leq |\mathcal{L}_i| + \aleph_0$

<u>this just means</u>: if $\mathcal{L}$ is finite or ctbl, then $\mathcal{L}^*$ is ctbl

if $\mathcal{L}$ is unctbl, then $|\mathcal{L}^*| = |\mathcal{L}|$

<u>Why</u>: to pass from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$
add symbol c_φ for every $\mathcal{L}_i$-fmla $\varphi(v)$

# of such fmlas $\leq$ # of finite sequences on the symbols from $\mathcal{L}_i \cup \{\neg, \wedge, \exists, =, (,), v_1, v_2, \ldots\}$ ctbl

which $= |\mathcal{L}_i|$ if $\mathcal{L}_i$ infinite

o/w $= \aleph_0$

So: either $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0 + \aleph_0 + \ldots = \aleph_0$ ← if $\mathcal{L}$ ctbl

or $|\mathcal{L}^*| = |\mathcal{L}| + |\mathcal{L}| + \ldots = |\mathcal{L}|$

← if $\mathcal{L}$ unctbl. ✓

So: we can expand $\mathcal{L}$ and T to get witness prop.; now we aim to make T max'l.

<u>Lemma</u> Spse T is a fin. sat. $\mathcal{L}$-theory and φ is a sentence. Then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is fin. sat.

not

Pf.: - Sps $T \cup \{\varphi\}$ is ~~not~~ fin. sat.
- then there is $\Delta \subseteq T \cup \{\varphi\}$ finite which has no model; must have $\varphi \in \Delta$ since T fin. sat. let $\Delta_0 = \Delta \setminus \{\varphi\}$
- must have $\Delta_0 \models \neg\varphi$.
- We claim $T \cup \{\neg\varphi\}$ is fin. sat.
- suffices to show: if $\Sigma \subseteq T$ is finite then $\Sigma \cup \{\neg\varphi\}$ has a model.
- Since $\Sigma \cup \Delta_0 \subseteq T$ is finite and T is fin sat. there is $M \models \Sigma \cup \Delta_0$
- since $\Delta_0 \models \neg\varphi$, $M \models \neg\varphi$.
- Thus $M \models \Sigma \cup \{\neg\varphi\}$ ✓

By iteratively adding φ or $\neg\varphi$ to T for every sentence φ, can extend a fin sat. T to a max'l fin. sat. T'.

Prop'n 2 if T is fin. sat. $\mathcal{L}$-theory, there is a maximal fin. sat. $\mathcal{L}$-theory $T' \supseteq T$.

Pf: need some form of Axiom of Choice
— we'll use Zorn's Lemma
- let I be set of all fin. sat. theories that contain T, ordered by inclusion $\subseteq$
- if $C \subseteq I$ is a chain of such theory define $T_C = \bigcup\{\Sigma : \Sigma \in C\}$

- We check T_C is fin. sat.
- if $\Delta \subseteq T_C$ is finite, there is $\Sigma \in C$ s.t. $\Delta \subseteq \Sigma$
  (Why? Two cases: C has a max el't, or not)
- Since Σ is fin. sat. there is $M \models \Delta$
- Thus T_C is fin. sat, i.e. $T_C \in I$
- T_C clearly an upper bound for C in I

$\Rightarrow$ can apply Zorn's lemma to find a maximal-under-inclusion $T' \in I$

- for any sentence φ, either $T' \cup \{\varphi\}$ or $T' \cup \{\neg\varphi\}$ fin. sat. by lemma

$\Rightarrow$ since T' max'l-under-$\subseteq$, either $\varphi \in T'$ or $\neg\varphi \in T'$
i.e. T' is maximal ✓

We can finally conclude:

<u>Theorem</u> (Compactness theorem) if T is a fin. sat. $\mathcal{L}$-theory then there is a model M of T

Further: if κ an infinite cardinal of size $\geq |\mathcal{L}|$, can arrange $|M| \leq \kappa$.

<u>Pf</u>- Prop'n 1 gives $\mathcal{L}^* \supseteq \mathcal{L}$ and fin sat $T^* \supseteq T$ s.t. any $\mathcal{L}^*$-theory extending T^* has witness prop.

- Prop. 2 gives max'l fin sat $T' \supseteq T^*$
- thm from last time gives $M \models T'$ (and hence $M \models T$) w/ $|M| \le |\mathcal{L}^*|$ ✓

<u>Ex's</u> ① <u>Claim</u> The class of torsion groups is not elementary, i.e. there is no theory T s.t. $M \models T$ iff M is a torsion-group.

Pf: - Sps there is such a T (in some lang $\mathcal{L}$ extending $\{\cdot, e\}$)
- let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ new constant symbol
- let φ_n be the sentence

$$\underbrace{c \cdot c \cdot \ldots \cdot c}_{n\text{-times}} \neq e$$

- <u>Claim</u>: $T \cup \{\varphi_1, \varphi_2, \ldots\} = T'$ is fin. sat.
  - fix $\Delta \subseteq T'$ finite
  - then $\Delta \subseteq T \cup \{\varphi_1, \ldots, \varphi_N\}$ for some large enough N.
  - let $M = (\mathbb{Z}_{N+1}, +, \cdot, c^M, \cancel{0, \ldots})$ zero = identity =1 be the cyclic group of order $N+1$ where we interpret c as generator !
  - then $M \models \Delta$ since $M \models T$ and $M \models \varphi_i$ for $i \le N$.
  - hence T' is fin sat.
  - By compactness, there is $G \models T'$

- But then G is a torsion group (since $G \models T$) w/ an element c^G of infinite order. (since $G \models \{\varphi_1, \varphi_2, \ldots\}$), a contradiction ✓

(2) - Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ and let $\mathcal{N} = (\mathbb{N}, +, \cdot, <, 0, 1)$ be the "standard model of arithmetic"
- Let $Th(\mathcal{N})$ be the full $\mathcal{L}$-theory of $\mathcal{N}$.

Claim There is an $\mathcal{L}$-structure $\mathcal{M} \models Th(\mathcal{N})$ with $a \in M$ s.t. a is bigger than every natural number.

Pf. first let's make sense of claim

- $Th(\mathcal{N})$ contains following sentences (since they are true in $\mathcal{N}$):

  $\forall x\ (x \not< 0)$ "0 is least element"

  $\forall x\ (x < x+1 \wedge \neg \exists y\, (x<y<x+1))$

  "every x has a successor $x+1$"

  "Every $x \neq 0$ has a predecessor"

  "$<$ is a linear order" ... etc.

- Thus if $\mathcal{M} \models Th(\mathcal{N})$ then $\mathcal{M}$ contains a copy of $\mathcal{N}$: i.e.