get $ex = ey$
- from axiom ③ $x = y$ ✓

- This informal proof can be turned into a formal one
- once "formal proof" is formalized can prove following (very reasonable) theorem:

Thm (Soundness of first-order logic)
if $T \vdash \varphi$ then $T \models \varphi$

Gödel showed the converse is true!

Thm (Completeness of F.O.L.)
if $T \models \varphi$ then $T \vdash \varphi$

a priori a challenge to prove:
from knowledge $\varphi$ is true in every model of $T$
need to pull out of air an actual proof of $\varphi$!

Def'n $T$ is inconsistent if for some sentence $\varphi$ we have $T \vdash \varphi \wedge \neg\varphi$

Corollary $T$ is consistent iff $T$ is satisfiable

Pf ($\Leftarrow$) if there is $\mathcal{M} \models T$ then $T$ is consistent since o/w by soundness there is $\varphi$ s.t. $\mathcal{M} \models \varphi \wedge \neg\varphi$, contradiction

($\Rightarrow$) if $T$ has no models then (trivially) $T \models \varphi \wedge \neg\varphi$ for some (actually every) $\varphi$.

But then $T \vdash \varphi \wedge \neg\varphi$ by completeness!

Corollary (The compactness theorem) a theory $T$ is satisfiable iff every finite subset $\Delta \subseteq T$ is satisfiable

Pf. ($\Rightarrow$) if $\mathcal{M} \models T$ then for any finite $\Delta \subseteq T$ we have $\mathcal{M} \models \Delta$

($\Leftarrow$) - if no such $\mathcal{M}$ then $T \vdash \varphi \wedge \neg\varphi$ for some $\varphi$, by prev. corollary

- since proofs are finite there is a finite $\Delta \subseteq T$ sufficient to prove $\varphi \wedge \neg\varphi$

$\Rightarrow$ $\Delta$ is unsatisfiable. ✓

Compactness Theorem looks innocent, but it is very powerful.

Ex (Finiteness is not first-order axiomatizable)

Fix $\mathcal{L}$. Sps $T$ is an $\mathcal{L}$-theory s.t. there are arbitrarily large

finite models of $T$. (i.e. $\forall n \in \mathbb{N}$ there is $\mathcal{M} \models T$ s.t. $|\mathcal{M}|$ finite but $> n$)
Then there is an infinite model of $T$.

Pf: For $n \geq 2$ let $\varphi_n$ be

$$\exists x_1 \ldots \exists x_n \left( \bigwedge_{i < j \leq n} x_i \neq x_j \right)$$

- Let $T' = T \cup \{\varphi_2, \varphi_3, \ldots\}$
- For any finite $\Delta \subseteq T'$, there is $N \in \mathbb{N}$ s.t. for all $n > N$ $\varphi_n \notin \Delta$
- Fix $\Delta$ and let $\mathcal{M} \models T$ w/ $|\mathcal{M}| > N$ (such $\mathcal{M}$ exists by hyp.)
- but then $\mathcal{M} \models \Delta$

$\Rightarrow$ $T'$ is finitely satisfiable

$\Rightarrow$ (by compactness) there is $\mathcal{M}$ s.t. $\mathcal{M} \models T'$

- but then $|\mathcal{M}|$ must be infinite and since $T \subseteq T'$, have $\mathcal{M} \models T$ too ✓

↳ proof of compactness thm above depends on syntactic notion of completeness

↳ would be nice to have a semantic proof, i.e. one where we really build a model!

# Henkin Models

- $T$ is finitely satisfiable if every finite $\Delta \subseteq T$ has a model
- Given a lang $\mathcal{L}$ and fin. sat'ble $T$, we build a model of (all of) $T$.

- construction has 2 parts:
  (1) we extend ~~[illegible]~~ $\mathcal{L}$ so it has enough constants to "witness every existential statement implied by $T$" and extend $T$ as much as consistently possible.
  (2) we let universe $M$ of our model $\mathcal{M}$ consist of the constant symbols in $\mathcal{L}$!
  ↳ not quite... issue is: for distinct $c, d \in \mathcal{L}$ might have $T \models c = d$, or even $c = d \in T$!
  ↳ so: if we want $\mathcal{M} \models T$ need to identify $c, d$ in our universe.
  ↳ so: we do this for all such $c, d$ and let $M$ be resulting set
  ... this works!

- We'll assume (1) and do (2) first then show (1) can be done
- construction requires a touch of set theory: just what a cardinal is and Zorn's Lemma
- ... read appendix A in book.

Def'n Given $\mathcal{L}$, and $\mathcal{L}$-theory $T$, say $T$ has the witness property if for every $\mathcal{L}$-fmla $\varphi(v)$ with one free var. there is a constant symbol $c \in \mathcal{L}$ s.t. $T \models (\exists v\, \varphi(v)) \Rightarrow \varphi(c)$

Def'n A $\mathcal{L}$-theory $T$ is maximal if for every $\mathcal{L}$-sentence $\varphi$, either $\varphi \in T$ or $\neg\varphi \in T$.

Lemma Sps $T$ is a maximal and finitely satisfiable theory. If $\Delta \subseteq T$ is finite and $\Delta \models \varphi$, then $\varphi \in T$.
for some sentence $\varphi$.

Pf. if $\varphi \notin T$ then $\neg\varphi \in T$ by maximality. But then $\Delta \cup \{\neg\varphi\} \subseteq T$ is a finite subset of $T$ w/ no model, contradiction fin. satisfiability of $T$.

Theorem Sps $T$ is a maximal and finitely satisfiable $\mathcal{L}$-theory w/ the witness property. Then: $T$ has a model.
In fact: there is $\mathcal{M} \models T$ w/ $|M| \leq \kappa$, where $\kappa$ = # of constant symbols in $\mathcal{L}$.

PF. - let $\mathcal{C}$ denote set of const. symbols in $\Sigma$
- define $c \sim d$ iff $c = d \in T$

Claim $\sim$ is an equiv. relation on $\mathcal{C}$.
PF: (refl.) given $c \in \mathcal{C}$, must have $c \sim c$ since $c = c \in T$ since o/w $c \neq c \in T$ (by max'ity) and then $T$ not fin. sat.
(sym.) if $c = d \in T$ then $d = c \in T$ by fin. sat. thus $c \sim d$ iff $d \sim c$.
(trans.) if $c = d$ and $d = e$ both $\in T$ then $c = e \in T$ since o.w. $c \neq e \in T$ and $T$ not fin. sat. Thus $c \sim d \sim e \Rightarrow c \sim e$ ✓

- let $M = \mathcal{C}/\sim$ be the set of equiv. classes of constants under $\sim$
- will be universe of our model $\mathcal{M}$
- for each $c \in \mathcal{C}$, let $c^* =$ equiv class of $c$ (so: $c^* = d^*$ iff $c \sim d$)
- so: $M = \{c^* : c \in \mathcal{C}\}$

↪ will interpret rln and function symbols in $\mathcal{M}$ in natural way ... just need to check it makes sense.

- Sps $R \in \mathcal{L}$ is n-ary rel'n symbol
- Sps $c_1, \dots, c_n, d_1, \dots, d_n \in \mathcal{C}$ and $c_i \sim d_i$ for all $i \le n$.
- then $R(c_1, \dots, c_n) \in T$ iff $R(d_1, \dots, d_n) \in T$ b/c T not fin. sat.

$\Rightarrow$ if we let $R^{\mathcal{M}} = \{(c_1^*, \dots, c_n^*) : R(c_1, \dots, c_n) \in T\}$ then $R^{\mathcal{M}}$ is well-defined.

- Now sps $f$ is n-ary func. symbol, $c_1, \dots, c_n \in \mathcal{C}$.
- Since $T \models \exists v\, f(c_1, \dots, c_n) = v$, by witness prop + maximality of $T$ there is $c$ s.t. $f(c_1, \dots, c_n) = c \in T$
- if $d_1, \dots, d_n, d \in \mathcal{C}$ and and $c_i \sim d_i$ and $c \sim d$ then $f(d_1, \dots, d_n) = d \in T$ too

$\Rightarrow$ can define $f^{\mathcal{M}}(c_1^*, \dots, c_n^*) = c^*$ iff $f(c_1, \dots, c_n) = c \in T$

↳ we've defined an $\mathcal{L}$-structure $\mathcal{M}$

↳ WTS: $\mathcal{M} \models T$

↳ first: show terms behave as expected

<u>Claim</u>: Given $t(v_1, \dots, v_n)$ a term and $c_1, \dots, c_n, d \in \mathcal{C}$ we have $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$ iff $t(c_1, \dots, c_n) = d \in T$

Pf ($\Leftarrow$) induction on construction of $t$ (you try!)

($\Rightarrow$) Sps $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. wts: $t(\vec{c}) = d \in T$

- Have $T \models \exists v\ t(\vec{c}) = v$
- by witness prop. there is $e \in \mathcal{C}$ s.t. $t(\vec{c}) = e \in T$
- So by ($\Leftarrow$) direction $t^{\mathcal{M}}(\vec{c}^*) = e^*$
- So $e^* = d^*$, so $e \sim d$, so $e = d \in T$
- Since $T$ is fin. sat, $t(\vec{c}) = d \in T$ ✓

Claim For all $\mathcal{L}$-fmlas $\varphi(v_1, \dots, v_n)$ and $c_1, \dots, c_n \in \mathcal{C}$ we have

$$\mathcal{M} \models \varphi(\vec{c}^*) \quad \text{iff} \quad \varphi(\vec{c}) \in T$$

Pf: Induction on $\varphi$

- if $\varphi$ is $t_1(\vec{v}) = t_2(\vec{v})$

then $\mathcal{M} \models \varphi(\vec{c}^*)$ iff $t_1^{\mathcal{M}}(\vec{c}^*) = t_2^{\mathcal{M}}(\vec{c}^*)$

iff there is $d \in \mathcal{C}$ s.t. $t_1^{\mathcal{M}}(\vec{c}^*) = t_2^{\mathcal{M}}(\vec{c}^*) = d^*$

iff (by claim) $t_1(\vec{c}) = d$ and $t_2(\vec{c}) = d$ both $\in T$

iff (by lemma) $t_1(\vec{c}) = t_2(\vec{c}) \in T$

i.e. iff $\varphi(\vec{c}) \in T$ ✓

- if $\varphi$ is $R(t_1(\vec{v}), \dots, t_n(\vec{v}))$

then $\mathcal{M} \models \varphi(\vec{c}^*)$ iff $(t_1^{\mathcal{M}}(\vec{c}^*), \dots, t_n^{\mathcal{M}}(\vec{c}^*)) \in R^{\mathcal{M}}$

iff $(d_1^*, \dots, d_n^*) \in R^{\mathcal{M}}$ where $d_i^* = t_i^{\mathcal{M}}(\vec{c}^*)$

iff $R(d_1, \dots, d_n) \in T$ (by prev. prev. claim)

iff (by lemma) $R(t_1(\vec{c}^*), \dots, t_n(\vec{c}^*)) \in T$

i.e. $\varphi(\vec{c}^*) \in T$ ✓

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- Sps claim true for $\varphi, \chi$

- if $\mathcal{M} \models \neg\varphi(\vec{c}^*)$ then $\mathcal{M} \not\models \varphi(\vec{c}^*)$
  so $\varphi(\vec{c}) \notin T$ by hypothesis
  so $\neg\varphi(\vec{c}) \in T$ by max'ity of $T$
- Conversely if $\neg\varphi(\vec{c}) \in T$ then
  $\varphi(\vec{c}) \notin T$ by fin. sat.'ability of $T$
  so $\mathcal{M} \not\models \varphi(\vec{c}^*)$ by i.h. so $\mathcal{M} \models \neg\varphi(\vec{c}^*)$

- Next, $\mathcal{M} \models \varphi \wedge \chi\ (\vec{c}^*)$ iff
  $\mathcal{M} \models \varphi(\vec{c}^*)$ and $\mathcal{M} \models \chi(\vec{c}^*)$
  iff (by IH) $\varphi(\vec{c})$ and $\psi(\vec{c}) \in T$
  iff (by lemma) $\varphi \wedge \chi(\vec{c}) \in T$

- Finally, sps claim true for $\varphi(\vec{v}, w)$
  and consider $\exists w\, \varphi(\vec{v}, w)$
- if $\mathcal{M} \models \exists w\, \varphi(\vec{c}^*, w)$ then
  there is $d^*$ s.t. $\mathcal{M} \models \varphi(\vec{c}^*, d^*)$
- by i.h. $\varphi(\vec{c}, d) \in T$ and so $\exists w\, \varphi(\vec{c}, w) \in T$
  by lemma

- Conversely if $\exists w\, \varphi(\vec{c}, w) \in T$ then
  by witness property there is $d$ s.t.
  $\varphi(\vec{c}, d) \in T$
- by i.h. $\mathcal{M} \models \varphi(\vec{c}^*, d^*)$
  so $\mathcal{M} \models \exists w\, \varphi(\vec{c}^*, w)$ ✓

- thus completes the induction
- we've proved in particular $M \models T$
- observe: by our construction, $|M| \leq |\mathcal{C}|$ ✓