<u>Prop'n</u>: Sps $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. If $X \subseteq M^n$ is A-definable and $j: M \to M$ is an automorphism of $\mathcal{M}$ that fixes A pointwise, then j fixes X setwise

i.e. if $j(a) = a$ for all $a \in A$
then for any $\vec{b} = (b_1, .., b_n) \in M^n$, have $\vec{b} \in X$ iff $j(\vec{b}) \in X$
$(j(b_1), .., j(b_n))$

"j restricts to a permutation of X"

<u>Pf</u>: - Sps $\varphi(\vec{v}, \vec{a})$ defines X over A
i.e. $\vec{a} \in A$ and $X = \{\vec{b} : \mathcal{M} \models \varphi(\vec{b}, \vec{a})\}$
- By prev. thm., since $j: \mathcal{M} \to \mathcal{M}$ is an isomorphism (i.e. automorphism) we have, for any $\vec{c} \in M$ that $\mathcal{M} \models \varphi(\vec{c})$ iff $\mathcal{M} \models \varphi(j(\vec{c}))$.
- in particular $\mathcal{M} \models \varphi(\vec{b}, \vec{a})$ iff
$\mathcal{M} \models \varphi(j(\vec{b}), j(\vec{a}))$ iff
$\mathcal{M} \models \varphi(j(\vec{b}), \vec{a})$

- but this says: $\vec{b} \in X$ iff $j(\vec{b}) \in X$, as desired.

Ex's

Claim

① if $\mathcal{M} = (M)$ is a pure set, then $X \subseteq M$ is definable <u>iff</u> X is finite or cofinite

pf: – we saw: finite/cofinite sets are definable

– So sps X is infinite and co-infinite

– Sps X were definable, say by $\varphi(x, \vec{a})$ for some $\vec{a} = (a_1, .., a_n) \in M^n$

– let $A = \{a_1, \ldots, a_n\}$

– Since both X and $X^c = M \setminus X$ infinite can find $x \in X$ and $y \in X^c$, s.t. $x, y \notin A$

– Consider the permutation $j : M \to M$ (which is an automorphism, since our lang. is empty) defined by:

$$j(x) = y$$
$$j(y) = x$$
$$j(m) = m \quad m \notin \{x, y\}$$

– then j fixes A pointwise, but $j[X] \neq X$, contradiction.

$\Rightarrow X$ not definable.

Q4: what are the definable subsets $X^2 \subseteq M^2$?

② Consider the rationals as a linear order $\mathcal{M} = (\mathbb{Q}, <)$

Claim: if $X \subseteq \mathbb{Q}$ is "infinite and co-infinite to the right"

(i.e. for every $q \in \mathbb{Q}$, there are $x \in X$, $y \in \mathbb{Q} \setminus X$ s.t. $x, y > q$)

then X is not definable

Pf: – Sps X were definable, say by $\varphi(x, \vec{a})$ for some $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Q}^n$

– by hypothesis, can find $x \in X$, $y \in \mathbb{Q} \setminus X$ s.t. both $x, y >$ all a_i's

– it will follow from our work later: there is an order-automorphism $j : (\mathbb{Q}, <) \to (\mathbb{Q}, <)$ s.t.

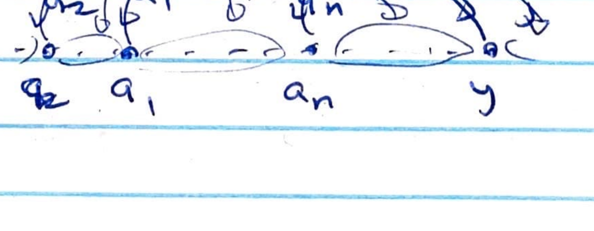
$$j(a_i) = a_i \quad \text{for every } i \leq n$$

$$j(x) = y$$

(– existence of j will follow from "ultrahomogeneity" of $(\mathbb{Q}, <)$)

– a contradiction! $j[X] \neq X$

$\Rightarrow X$ undefinable



③ <u>Claim</u> $2\mathbb{Z}$ is not $\emptyset$-definable in $(\mathbb{Z}, <)$

<u>Pf</u>:- Prop'n $\Rightarrow$ if X is $\emptyset$-definable then any automorphism $j : (\mathbb{Z}, <) \to (\mathbb{Z}, <)$ fixes X setwise
- Consider $j : \mathbb{Z} \to \mathbb{Z}$ $j(z) = z+1$
- $z < z' \Leftrightarrow j(z) < j(z') \Leftrightarrow z < z+1$
- j a bijection $\Rightarrow$ j an automorphism
- but $j[2\mathbb{Z}] \neq 2\mathbb{Z}$ (in fact $j[2\mathbb{Z}] \cap 2\mathbb{Z} = \emptyset$)
  $\Rightarrow$ $2\mathbb{Z}$ not $\emptyset$-definable

<u>Notice</u>: same arg shows: only nonempty $\emptyset$-definable subset of $(\mathbb{Z}, <)$ is $\mathbb{Z}$

... but is $2\mathbb{Z}$ definable in $(\mathbb{Z}, <)$ w/ parameters?
- if <u>not</u> Prop'n doesn't help: if $a \in \mathbb{Z}$ is any integer parameter and $j : \mathbb{Z} \to \mathbb{Z}$ an auto s.t. $j(a) = a$ then actually $j = \text{identity}$ (... why?)

$\hookrightarrow$ need a stronger analysis ...

Here is a (somewhat) concrete characterization of the definable sets:

<u>Theorem</u> Let $\mathcal{M}$ be an $\mathcal{L}$-structure. For each $n \geq 1$, let $\mathcal{D}_n$ be the collection of subsets of M^n s.t. $\mathcal{D} = \bigcup_{n \geq 1} \mathcal{D}_n$ is the smallest collection of sets w/ the following closure properties:

1. $M^n \in \mathcal{D}_n$
2. for all n-ary function symbols f, $\text{graph}(f^{\mathcal{M}}) = \{(x_1, \dots, x_n, y) \in M^{n+1} : f(x_1, \dots, x_n) = y\} \in \mathcal{D}_{n+1}$
3. for all n-ary rln symbols R, $R^{\mathcal{M}} \in \mathcal{D}_n$
4. for all $i, j \leq n$, $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\} \in \mathcal{D}_n$
5. if $X \in \mathcal{D}_n$ then $M \times X \in \mathcal{D}_{n+1}$
6. if $X, Y \in \mathcal{D}_n$, then all of $M^n \setminus X$, $X \cap Y$, and $X \cup Y$ are in $\mathcal{D}_n$
7. if $X \in \mathcal{D}_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection map $\pi(x_1, \dots, x_n, x_{n+1}) = \pi(x_1, \dots, x_n)$ then $\pi[X] \in \mathcal{D}_n$
8. if $X \in \mathcal{D}_{n+m}$ and $b \in M^m$ then $\{\vec{a} \in M^n : (\vec{a}, \vec{b}) \in X\} \in \mathcal{D}_n$

Then: for any $n \geq 1$, $X \subseteq M^n$ is definable iff $X \in \mathcal{D}_n$

Pf: - Let Def denote collection of definable subsets of M.
- WTS $Def = \mathcal{D}$
- For ($\supseteq$): suff. to show Def satisfies ①-⑧ (since $\mathcal{D}$ is smallest class sat'ing ①-⑧)

① M^n is definable by $\varphi(v_1,..,v_n)$: $v_1 = v_1$
② graph(f^M) def'able by $\varphi(v_1,..,v_n,y)$: $f(v_1,..,v_n) = y$
③ R^M def'able by fmla $R(v_1,..,v_n)$
④ $\{\vec{x} \in M^n : x_i = x_j\}$ def'able by $v_i = v_j$
⑤ if $X \subseteq M^n$ is def'able by $\varphi(v_1,..,v_n,\vec{a})$ then $M \times X$ def'able by $\varphi(v_2,..,v_n,\vec{a})$
⑥ if $X, Y \subseteq M^n$ defined by $\varphi(\vec{v},\vec{a})$ and $\chi(\vec{v},\vec{b})$ resp. then $\neg X, X \cap Y, X \cup Y$ defined by $\neg\varphi$, $\varphi \wedge \chi$, $\varphi \vee \chi$ respectively.
⑦ if $X \subseteq M^{n+1}$ def'd by $\varphi(v_1,..,v_n,v_{n+1},\vec{a})$ then $\pi[X]$ def'd by $\exists v_{n+1}\, \varphi(v_1,..,v_n\, v_{n+1},\vec{a})$
⑧ if $X \subseteq M^{n+m}$ def'd by $\varphi(v_1,..,v_n,v_{n+1},..,v_{n+m},\vec{c})$ and $\vec{b} \in M^m$, then $\{\vec{a} : (\vec{a},\vec{b}) \in X\}$ def'd by $\varphi(\vec{v},\vec{b},\vec{c})$

$\Rightarrow$ $\mathcal{D} \subseteq Def$ ✓

For reverse containment: WTS if X definable then $X \in \mathcal{D}$.

- will need:

<u>Claim</u> if $t(v_1, \dots, v_n)$ a term then
$\text{graph}(t^{\mathcal{M}}) = \{(x_1, \dots, x_n, y) \in M^{n+1} : t^{\mathcal{M}}(\vec{x}) = y\} \in \mathcal{D}_{n+1}$

<u>Pf</u>: - by induction on construction of t
- if t is c, think of $t^{\mathcal{M}}(\vec{v})$ as the constant function $t^{\mathcal{M}}(\vec{x}) = c^{\mathcal{M}}$
- so need to show $\{(\vec{x}, c^{\mathcal{M}}) : \vec{x} \in M^n\} \in \mathcal{D}_{n+1}$
  - ↳ by (4) $\{(c^{\mathcal{M}}, c^{\mathcal{M}})\} \in \mathcal{D}_2$
  - by (8) $\{c^{\mathcal{M}}\} \in \mathcal{D}_1$
  - by n applications of (5), $\{(\vec{x}, c^{\mathcal{M}}) : \vec{x} \in M^n\} \in \mathcal{D}_{n+1}$ ✓

- if t is v_i need to show $\{(\vec{x}, x_i)\} \in \mathcal{D}_{n+1}$
  - ↳ follows from (1) + (4)

- Sps claim is true for $t_1(\vec{v}), \dots, t_m(\vec{v})$ and sps t is $f(t_1, \dots, t_m)$
- for simplicity sps $m = 1$, so t is $f(t_1)$
- Let $G_1 = \text{graph}(t_1^{\mathcal{M}})$, so hyp is $G_1 \in \mathcal{D}_{n+1}$
- Let $G = \text{graph}(f^{\mathcal{M}})$ which is $\in \mathcal{D}_2$ by (2)
- then $\text{graph}(t^{\mathcal{M}})$ is:

$$\{(\vec{x}, y) : \text{there is } z \in M \text{ s.t. } t_1^{\mathcal{M}}(\vec{x}) = z \text{ and } f^{\mathcal{M}}(z) = y\}$$

which is the projection of

$$\{(\vec{x}, y, z) : t_1^{\mathcal{M}}(\vec{x}) = z \text{ and } f^{\mathcal{M}}(z) = y\}$$

which is the intersection of G_1 and G (or really $M^{n-2} \times G$)

$\hookrightarrow$ follows from 7, 6, 5 that $\mathrm{graph}(f^{\mathcal{M}}) \in \mathcal{D}_{n+1}$ ✓

Next we prove

Claim for every $n \geq 1$ and for every $\emptyset$-definable $X \subseteq M^n$ we have $X \in \mathcal{D}_n$

Pf: Sps X def'd by $\varphi(\vec{v})$, we induct on construction of φ.

- if φ is $t_1 = t_2$ then

$X = \{\vec{x} \in M^n : t_1^{\mathcal{M}}(\vec{x}) = t_2^{\mathcal{M}}(\vec{x})\}$

$= \{\vec{x} :$ there are $y, z \in M$ s.t. $t_1^{\mathcal{M}}(\vec{x}) = y$ and $t_2^{\mathcal{M}}(\vec{x}) = z$ and $y = z\}$

which is $\in \mathcal{D}_n$ using prev. claim and closure properties of $\mathcal{D}$

- if φ is $R(t_1, \dots, t_m)$ then

$X = \{\vec{x} \in M^n : (t_1^{\mathcal{M}}(\vec{x}), \dots, t_m^{\mathcal{M}}(\vec{x})) \in R^{\mathcal{M}}\}$

$= \{\vec{x} : \exists z_1, \dots, z_m$ s.t. $t_1^{\mathcal{M}}(\vec{x}) = z_1 \wedge \dots \wedge t_m^{\mathcal{M}}(\vec{x}) = z_m \wedge (z_1, \dots, z_m) \in R^{\mathcal{M}}\}$

which is $\in \mathcal{D}_n$ by closure (u try..)

- Thus all $\emptyset$-def'ble sets X that are def'd by <u>atomic</u> fmlas are in $\mathcal{D}$
- Since $\mathcal{D}$ closed under complements, $\cup$, $\cap$ and projections follows that all $\emptyset$-def sets X def'd by fmlas built from atomic ones using $\neg, \wedge, \vee, \exists$ are in $\mathcal{D}$

  i.e., <u>all</u> $\emptyset$-def sets X. ✓

- then (*) + claim $\Rightarrow$ all def'able sets in $\mathcal{D}$ ✓

- Theorem hard to use in practice: e.g. to know which $X \subseteq M$ are definable need to know which $Y \subseteq M^n$ are definable already, for all $n \geq 1$.

<u>The Compactness Theorem</u>

<u>Def'n</u> Given $\mathcal{L}$, an $\mathcal{L}$-theory T, and a sentence φ

say φ is a <u>logical consequence</u> of T

write $T \models \varphi$

if whenever $\mathcal{M} \models T$ then $\mathcal{M} \models \varphi$

e.g. if T is the axioms for groups and φ is $\forall x \forall y \forall g\ (g \cdot x = g \cdot y \Rightarrow x = y)$ then $T \models \varphi$ since φ is true in every group $\mathcal{G} = (G, \circ, e)$

Def'n Given a theory T and sentence φ

Say T proves φ

write $T \vdash \varphi$

if there is a (formal) proof of φ from T

↳ won't formally define "formal proof from T" but it can be done!

↳ Int'n: a formal proof is a finite sequence of sentences, each of which is either in T or follows from previous ones according to simple deduction rules like "from $\varphi \Rightarrow \psi$ and φ, deduce ψ" "from $\forall x\, \varphi(x)$, deduce $\varphi(c)$"

E.g. if T is group axioms and φ is $\forall x \forall y \forall g\ (g \cdot x = gy \Rightarrow x = y)$ again then $T \vdash \varphi$ since there is a pf of φ from T, e.g.:

- fix x, y, g
- by axiom ② from T: $\exists h$ s.t. $hg = e$
- thus $h(gx) = h(gy)$ so by associativity (i.e. axiom ①)