Converse to thm not true: $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ but not $\cong$ (not even same cardinality!)

Theories

- given $\mathcal{L}$, an $\mathcal{L}$-theory is a set of $\mathcal{L}$-sentences
- given $\mathcal{L}$-structure $\mathcal{M}$, write
  $\mathcal{M} \models T$ if $\mathcal{M} \models \varphi$ for every $\varphi \in T$
- T is satisfiable if it has a model.
  i.e. there is $\mathcal{M}$ s.t. $\mathcal{M} \models T$

e.g. if $T = \{\varphi\}$ where φ: $\exists x(x=x)$
then T is sat'ble (every $\mathcal{M} \models T$)
but $T' = \{\exists x(x \neq x)\}$
is not.

Two approaches:
① Start w/ a structure $\mathcal{M}$, study its full theory, i.e. consider $T = Th(\mathcal{M})$
② start w/ T, study its models $\mathcal{M}$ ← often finite, i.e. a "list of axioms"

Def'n a class of structures K is called an elementary class if there is a theory T s.t. $\mathcal{M} \in K$ iff $\mathcal{M} \models T$.

Ex's of theories / elem. classes

① (Infinite sets) Let $\mathcal{L} = \emptyset$. For $n \in \mathbb{N}$ let φ_n denote:

$$\exists x_1 \cdots \exists x_n \left(\bigwedge_{i<j\leq n} x_i \neq x_j \right)$$

"there exist n distinct elements"

Let $T = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}$

For $\mathcal{M} = (M)$ an $\mathcal{L}$-struct (i.e. a set) have $\mathcal{M} \models T$ iff M is infinite

So: class K of inf. sets is elem.

② (Linear orders)

- Let $\mathcal{L} = \{<\}$ (really I mean: $\mathcal{L} = \{R\}$ binary)
- Let T be following sentences:

$$\forall x \, (x \not< x)$$

$$\forall x \forall y \forall z \left[(x<y \wedge y<z) \Rightarrow (x<z) \right]$$

$$\forall x \forall y \, (x<y \vee y<x \vee x=y)$$

Then $\mathcal{M} \models T$ iff $\mathcal{M} = (M, <)$ is a linear order.

③ (Groups) Let $\mathcal{L} = \{\cdot, e\}$

Let T be:

$$\forall x \forall y \forall z \; (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$\forall x \exists y \; (x \cdot y = e) \wedge (y \cdot x = e)$$

$$\forall x \; (x \cdot e = x) \wedge (ex = x)$$

Then $\mathcal{M} \models T$ iff $\mathcal{M} = (G, \circ, e)$ is a group.

4 (Torsion-free groups) Let $\mathcal{L} = \{\circ, e\}$ again. For $n \geq 1$ let φ_n be:

$$x \neq e \Rightarrow \underbrace{x \circ x \circ \cdots \circ x}_{n\text{-times}} \neq e$$

Let T be theory from 3 and let $T' = T \cup \{\varphi_1, \varphi_2, \ldots\}$

Then $\mathcal{M} \models T'$ iff $\mathcal{M} = (G, \circ, e)$ is a <u>torsion-free group</u>

<u>Def'n</u> a group G is <u>torsion</u> if for every $g \in G$ there is $n \in \mathbb{N}$ s.t. $g^n = \underbrace{g \cdot g \cdots g}_{n\text{-times}} = e$.

↳ all finite groups are torsion, but there are infinite torsion groups too

e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

- intuitively "G is torsion" not a 1st-order statement, would have to write:

$$\forall g \, (g = e \vee g \cdot g = e \vee g \cdot g \cdot g = e \vee \ldots)$$

- intuition is correct: class of torsion groups is <u>not</u> elementary or sometimes (slightly weaker) def'n every bounded A

Def'n A linear order $(L, <)$ is **complete** if every $A \subseteq L$ has both a least upper bound and greatest lower bound.

e.g. $(\mathbb{R}, <)$ complete $(\mathbb{Q}, <)$ not

- Intuition: completeness not a first-order property – refers to subsets of L not just elements
- is correct: class of complete linear orders is not elementary

Other examples of elementary classes:

- rings
- fields
- graphs
- ordered groups
- posets
- equiv. relations

... etc.

Definability

- Definability is an important concept in many subfields of math'l logic.
- why? Definable sets often "behave nicely" compared to arbitrary sets
- Int'n: a set X is definable if it can be described in language we have access to
- In first-order logic context this means: described by a first-order fmla

Ex's

① - Consider group $\mathcal{Z} = (\mathbb{Z}, +, 0)$
- Consider fmla $\varphi(x)$:

$$\exists y\,(y+y=x)$$

- Given $n \in \mathbb{Z}$, we have

$$\mathcal{Z} \models \varphi(n) \text{ iff } n \text{ is even}$$

i.e. $\{n \in \mathbb{Z} : \mathcal{Z} \models \varphi(n)\} = \{\ldots -2, 0, 2, 4, \ldots\}$
$= \text{"}2\mathbb{Z}\text{"}$

↳ we think of $\varphi(x)$ as being a definition of the set $2\mathbb{Z}$ of even integers

↳ say: $2\mathbb{Z}$ is definable in $\mathcal{Z}$

② - consider now the semigroup $\mathcal{M} = (\mathbb{Z}, \cdot)$
- $2\mathbb{Z}$ is also definable in $\mathcal{M}$, but now we

need a "parameter"

- Consider fmla $\varphi(x,y)$:
  $$\exists z\,(x = y \cdot z)$$ "x is a multiple of y"

- Then: $\{n \in \mathbb{Z} : \mathcal{M} \models \varphi(x, \overset{\text{two}}{2})\} = 2\mathbb{Z}$
  ← "x is a multiple of 2"

- still say: $2\mathbb{Z}$ is definable in $\mathcal{M}$
  (.. using the parameter 2)

Q: is $2\mathbb{Z}$ definable in $(\mathbb{Z}, <)$? Hm!

- Here's the general def'n:

Def'n - Sps $\mathcal{M}$ is an $\mathcal{L}$-structure. A subset $X \subseteq M^n$ is definable in $\mathcal{M}$ if there is an $\mathcal{L}$-fmla $\varphi(v_1, .., v_n, w_1, .., w_m)$ and a $\vec{b} = (b_1, .., b_m) \in M^m$ s.t.
$$X = \{\vec{a} \in M^n : \mathcal{M} \models \varphi(\vec{a}, \vec{b})\}$$
- if there is such φ, say: φ defines X

- Given $A \subseteq M$, say X is A-definable (or definable over A) if there is $\varphi(\vec{v}, \vec{w})$ and $\vec{b} \in A^m$ s.t. $X = \{\vec{a} \in M^n : \mathcal{M} \models \varphi(\vec{a}, \vec{b})\}$

(So: X is definable iff X is M-definable)

More ex's

① Sps $\mathcal{M} = (M, \ldots)$ is a structure and $A \subseteq M$ is finite

- then A is definable (over itself) in $\mathcal{M}$: $A = \{a_1, \ldots, a_n\}$
- Consider $\varphi(x, \vec{v}) := (x = v_1) \vee (x = v_2) \vee \ldots \vee (x = v_n)$

  so $\varphi(x, \vec{a}) := (x = a_1) \vee (x = a_2) \vee \ldots \vee (x = a_n)$
- cfc. $A = \{m \in M : \mathcal{M} \models \varphi(m, \vec{a})\}$

Symmetrically: if $B \subseteq M$ is cofinite, i.e. $B = M \setminus A$ for some finite $A = \{a_1, \ldots, a_n\} \subseteq M$, then B is definable too (over A): take negation of fmla above.

② More generally: if $X \subseteq M^n$ is definable w/ definition $\varphi(\vec{v}, \vec{b})$ for some parameters $\vec{b} \in M$, then the complement X^c of X is definable too, by $\neg\varphi(\vec{v}, \vec{b})$

↳ $\neg$ connective corresponds to complements on definable sets

↳ what about $\wedge$ and $\vee$?

or $\exists x$, $\forall x$?

(3) - Consider $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$ = field of reals.

- Let $\varphi(x,y)$ be:

$$\exists z \, (z \neq 0 \wedge y = x + z^2)$$

$\neg(z=0)$

- For $a, b \in \mathbb{R}$, have $\mathcal{M} \models \varphi(a,b)$ iff $a<b$

$\Longrightarrow$ So: order relation $<$ is $\emptyset$-definable in $\mathcal{M}$

$\Longrightarrow$ implies for most purposes, $(\mathbb{R}, +, \cdot, 0, 1)$ is "same" as $(\mathbb{R}, +, \cdot, <, 0, 1)$

(4) Consider $\mathcal{M} = (\mathbb{Z}, +, \cdot, 0, 1)$ = ring of integers

- is $<$ definable in $\mathcal{M}$? fmla above doesn't work

- <u>Lagrange's theorem</u>: every non-negative integer k can be written as the sum of four ~~squares~~ integer squares: $k = k_1^2 + k_2^2 + k_3^2 + k_4^2$

- Let $\varphi(x,y)$ be:

$$\exists z_1 \exists z_2 \exists z_3 \exists z_4 \, (z_1 \neq 0 \wedge y = x + z_1^2 + z_2^2 + z_3^2 + z_4^2)$$

- then $m<n$ (in $\mathbb{Z}$) iff $\mathcal{M} \models \varphi(m,n)$

- So $<$ is definable!

↳ the point: non-triv. theorems can influence which sets are definable!

... more ex's in book

Following theorem sometimes helps to prove certain sets are <u>not</u> definable

<u>Prop'n</u>: Sps $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. If $X \subseteq M^n$ is A-definable and $j: M \to M$ is an automorphism of $\mathcal{M}$ that fixes A pointwise, then j fixes X setwise

i.e. if $j(a) = a$ for all $a \in A$
then for any $\vec{b} = (b_1, \ldots, b_n) \in M^n$, have $\vec{b} \in X$ iff $\underbrace{j(\vec{b})}_{(j(b_1), \ldots, j(b_n))} \in X$

"j restricts to a permutation of X"

<u>Pf</u>: - Sps $\varphi(\vec{v}, \vec{a})$ defines X over A
i.e. $\vec{a} \in A$ and $X = \{\vec{b} : \mathcal{M} \models \varphi(\vec{b}, \vec{a})\}$

- By prev. thm., since $j: \mathcal{M} \to \mathcal{M}$ is an isomorphism (i.e. automorphism) we have, for any $\vec{c} \in M$ that $\mathcal{M} \models \varphi(\vec{c})$ iff $\mathcal{M} \models \varphi(j(\vec{c}))$.

- in particular $\mathcal{M} \models \varphi(\vec{b}, \vec{a})$ iff
$\mathcal{M} \models \varphi(j(\vec{b}), j(\vec{a}))$ iff
$\mathcal{M} \models \varphi(j(\vec{b}), \vec{a})$

- but this says: $\vec{b} \in X$ iff $j(\vec{b}) \in X$, as desired.